

A Universal Nonlinear Relation among Boundary States in Closed String Field Theory

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Abstract

We show that the boundary states satisfy a nonlinear relation (the idempotency equation) with respect to the star product of closed string field theory. This relation is universal in the sense that various D-branes, including the infinitesimally deformed ones, satisfy the same equation, *including* the coefficient. This paper generalizes our analysis (hep-th/0306189) in the following senses. (1) We present a background-independent formulation based on conformal field theory. It illuminates the geometric nature of the relation and allows us to more systematically analyze the variations around the D-brane background. (2) We show that the Witten-type star product satisfies a similar relation but with a more divergent coefficient. (3) We determine the coefficient of the relation analytically. The result shows that the α parameter can be formally factored out, and the relation becomes universal. We present a conjecture on vacuum theory based on this computation.

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§1. Introduction

In quantum field theories, soliton solutions play an essential role in providing an understanding of the non-perturbative properties of the system. Well-known examples are the instanton and monopole solutions in non-abelian gauge theories and the black hole solutions in Einstein gravity. They are exact solutions of nonlinear equations that reflect the topological structure of the configuration space. In string theory, the corresponding objects are D-branes and the NS 5-brane. The discoveries of these solutions have led to major breakthroughs in the history of string theory.

Among these solitons, D-branes¹⁾ are known to have completely stringy descriptions in terms of boundary states.²⁾ They include all the effects of the massive modes of the closed string in a compact form. A natural question is whether D-branes can be understood as solutions of a nonlinear equation of string field theory (SFT), as in particle theory.

Such a question was originally posed in the context of open string field theory.³⁾ In particular, in the proposal of vacuum string field theory (VSFT)⁴⁾ (see also the partial list of Refs. 5)–8) for related studies), it is conjectured that D-branes are described by projectors of Witten’s star product. This idea has stimulated widespread interest in string field theory. It seems to be a natural idea in the context of noncommutative geometry, where D-branes are described by noncommutative solitons, namely the projection operators.⁹⁾

There exists, however, a technical challenge in this scenario. The boundary state belongs to the closed string Hilbert space. In order to describe it with the *open* string variable, it is necessary to use singular states, such as sliver or butterfly states. The treatment of these states is usually subtle, and one needs a careful analysis including a regularization scheme.¹

In a previous paper,¹⁰⁾ we proposed to look at this problem from a different angle. We used closed string field theory directly and attempted to derive nonlinear relations that are satisfied by the boundary states. In particular, we examined the star product for a closed string (the covariant version of the light-cone vertex) studied by Hata, Itoh, Kugo, Kunitomo and Ogawa (HIKKO)¹¹⁾ and proved that boundary states satisfy the idempotency equation

$$|B\rangle \star |B\rangle \propto |B\rangle, \quad (1.1)$$

which is strikingly similar to the conjecture⁴⁾ in the open string case. We have confirmed the validity of this identity for a large family of boundary states, including those with constant electro-magnetic flux. Furthermore, variations of this equation seem to give proper on-shell conditions of the open string mode on the D-brane.

¹ See Ref. 8) for the recent studies based on butterfly states with a regularized treatment of the mid-point singularity.

The computation in Ref. 10) is based on the oscillator formulation.¹¹⁾ That computation has the definite merit that it is reduced to straightforward algebraic manipulations. At the same time, however, it has certain drawbacks, such as divergence resulting from the infinite dimensionality of the Neumann matrices. It also depends explicitly on the flat background where the closed string oscillator is defined.

In this paper, we use an alternative formulation of string field theory based on conformal field theory. This is the formulation which was developed by LeClair, Peskin and Preitskopf (LPP).¹²⁾ It sheds light on some issues that are obscure in the oscillator formulation. For example, because only the properties of the stress-energy tensor are used, it does not depend on the particular representation of the background CFT. It also provides a clear geometrical picture, and sometimes we can define a natural prescription of the regularization in which the oscillator calculation becomes ambiguous due to divergences.

In particular, we are able to prove the idempotency relation in a background-independent fashion. We also derive some explicit results answering questions that were left unanswered in the previous paper. Specifically, we compute (a) the normalization factor in the relation (1.1). We show in particular that the unphysical α parameters can be factored out of this relation. As an important corollary, we show that the idempotency relation is *universal*, including the coefficient for any boundary states in the flat background. This result is essential to develop a possible vacuum theory. (b) For the deformation of the boundary state through a vector-type variation, which corresponds to a massless photon on the D-brane, we show that transversality is implied from (1.1). We could not reach this conclusion in Ref. 10), as it contains a term of the form $0 \times \infty$ in the coefficient, making it difficult to treat. (c) We examine the star product using a Witten-type three string vertex for the closed string.¹³⁾ We show, in particular, that the basic relation (1.1) remains valid while the coefficient in the relation diverges very strongly, as $\delta(0)^\infty$. This divergence occurs whenever we consider an overlap of the boundary states. We note that a similar divergence appears in the inner product of boundary states. In any case, our analysis shows that the idempotency relation is valid for both HIKKO and Witten type vertices. This is another sort of the universality of the idempotency relation.

We organize the paper as follows. In §2, we give a review of the HIKKO theory in LPP language. In §3, we re-express the main statements of Ref. 10) in this formulation. We also include several new results, which are proved in the later sections. Readers who are unfamiliar with Ref. 10) may wish to read this section first to understand the overall picture of our results. In §4, we derive the relation (1.1) for a Witten-type vertex, which is used in nonpolynomial closed string field theory.^{13)–16)} In §5, we give a background-independent proof of the idempotency relation in the conformal field theory picture. In §6, we show that

the solutions to the infinitesimal deformation of the relation around a boundary state can be identified with the open string spectrum on the D-brane. In the appendices, we explain our notation and the correspondence with our previous paper. We also give an explicit computation of the coefficient appearing in the star product.

§2. CFT description of closed string field theory

2.1. String vertices from conformal mapping

In string field theory, the N -string vertex $\langle v_N|$, which specifies the interactions of N strings, is the fundamental object to construct the action. $\langle v_N|$ is a mapping from N string Hilbert space to the set of complex numbers: $\mathcal{H}^{\otimes N} \rightarrow \mathbf{C}$. LeClair, Peskin and Preitskopf (LPP)¹²⁾ defined $\langle v_N|$ in terms of N conformal mappings $h_r(w_r)$ (with $r = 1, \dots, N$) from N disks with coordinates w_r to a Riemann sphere Σ . For each element in the Hilbert space $|A_r\rangle \in \mathcal{H}$, $r = 1, \dots, N$, we denote by $\mathcal{O}_{A_r}(w_r)$ the corresponding operator defined through $|A_r\rangle = \mathcal{O}_{A_r}(0)|0\rangle$. LeClair, Peskin and Preitskopf defined $\langle v_N|$ using these data as

$$\langle v_N|A_1\rangle|A_2\rangle\cdots|A_N\rangle = \langle h_1[\mathcal{O}_{A_1}] h_2[\mathcal{O}_{A_2}] \cdots h_N[\mathcal{O}_{A_N}] \rangle, \quad (2.1)$$

where the right-hand side is a correlation function of conformal field theory (CFT) on Σ . Here, $h_r[\mathcal{O}_{A_r}]$ is an operator on Σ defined by applying the conformal transformation h_r to the operator \mathcal{O}_{A_r} . If $\mathcal{O}(w_r)$ is a primary field of conformal dimension h , the image of the mapping is $h_r[\mathcal{O}(0)] = (dh_r(w_r)/dw_r)^h \mathcal{O}(h_r(w_r))|_{w_r=0}$. The anti-holomorphic part is given in the same way.

In the HIKKO formulation of closed string field theory, there is an extra parameter (the α -parameter), which represents the length of the closed string at the interaction point. This is analogous to the light-cone momentum p^+ in the light-cone string field theory.^{17)–19)} It is additively conserved during the entire process. We need to include the dependence on α in the string field, and we make this explicit by writing

$$|\Phi(\alpha)\rangle = |\Phi\rangle \otimes |\alpha\rangle, \quad (2.2)$$

where the ket vector $|\alpha\rangle$ is the eigenvector of the operator $\hat{\alpha}$ with eigenvalue α . The normalization of eigenstates is given by $\langle\alpha_1|\alpha_2\rangle = 2\pi\delta(\alpha_1 - \alpha_2)$. The other factor, $|\Phi\rangle$, is an element of the conventional string field, which can be expanded as

$$|\Phi\rangle = \sum_A \mathcal{O}_A|0\rangle\phi_A, \quad (2.3)$$

where ϕ_A is a component field and $\mathcal{O}_A|0\rangle$ is a closed string state.

We define two kinds of products between string fields, the dot (\cdot) product and the star (\star) product, using CFT language. The dot product of two string fields yields a complex number (i.e., $\mathcal{H} \otimes \mathcal{H} \rightarrow \mathbf{C}$):

$$\begin{aligned}\Phi_1(\alpha_1) \cdot \Phi_2(\alpha_2) &\equiv -2\pi\delta(\alpha_1 + \alpha_2)(-1)^{|\Phi_1|} \langle \Phi_1 | b_0^- | \Phi_2 \rangle \\ &\equiv -2\pi\delta(\alpha_1 + \alpha_2)(-1)^{|\Phi_1|} \langle I[\Phi_1] | b_0^- | \Phi_2 \rangle\end{aligned}\quad (2.4)$$

$$= (-1)^{|\Phi_1||\Phi_2|} \Phi_2(\alpha_2) \cdot \Phi_1(\alpha_1), \quad (2.5)$$

where $I(z) = 1/z$ is the inversion map, $\langle \cdots \rangle$ represents a correlator of CFT on Σ and $(-1)^{|\Phi|}$ denotes the Grassmann parity of the string field Φ . In addition to the α -dependent factor, we insert b_0^- into the correlator as a convention. The last equality implies that the dot product satisfies the (graded) commutativity Eq. (2.5). We can rewrite Eq. (2.4) using the reflector $\langle \hat{R}(1, 2) |$ as

$$\Phi_1(\alpha_1) \cdot \Phi_2(\alpha_2) = \langle \hat{R}(1, 2) | b_0^{-(2)} | \Phi_1 \rangle_1 | \Phi_2 \rangle_2, \quad (2.6)$$

where $\langle \hat{R}(1, 2) |$ is obtained from the LPP reflector² (2-string vertex) $\langle R(1, 2) |$ given in Eq. (A.7):

$$\langle \hat{R}(1, 2) | = - \int \frac{d\alpha_1}{2\pi} \frac{d\alpha_2}{2\pi} {}_1\langle \alpha_1 | {}_2\langle \alpha_2 | \langle R(1, 2) | 2\pi\delta(\alpha_1 + \alpha_2). \quad (2.7)$$

The \star product, which determines the 3-string vertex, is defined by the CFT correlator,³ in combination with the dot product:

$$\begin{aligned}(\Phi_1(\alpha_1) \star \Phi_2(\alpha_2)) \cdot \Phi_3(\alpha_3) &= -2\pi\delta(\alpha_1 + \alpha_2 + \alpha_3)(-1)^{|\Phi_2|} \\ &\quad \times \langle h_1[b_0^- \wp \Phi_1] h_2[b_0^- \wp \Phi_2] h_3[b_0^- \wp \Phi_3] \rangle\end{aligned}\quad (2.8)$$

$$= (-1)^{|\Phi_1|(|\Phi_2|+|\Phi_3|)} (\Phi_2(\alpha_2) \star \Phi_3(\alpha_3)) \cdot \Phi_1(\alpha_1) \quad (2.9)$$

$$= \Phi_1(\alpha_1) \cdot (\Phi_2(\alpha_2) \star \Phi_3(\alpha_3)). \quad (2.10)$$

We note that an operator $b_0^- \wp$ is inserted into the correlator. The projector \wp is defined by

$$\wp = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{i\theta(L_0 - \tilde{L}_0)}, \quad (2.11)$$

where L_0 and \tilde{L}_0 are total (i.e., matter+ghost) Virasoro operator. It imposes the level matching condition on string fields.

² In the following, we use the term “the LPP vertex” in reference to the vertex operator defined by Eq. (2.1) without inclusion of the ghost zero-mode insertion and the state vectors for the α parameter.

³ In Ref. 11), the dot product and the star product are defined using purely oscillator language, and their explicit correspondence to the LPP language can be understood from Ref. 20). We summarize this correspondence in Appendix A. Note that our convention is different from that used in Ref. 13).

The conformal mappings $h_r(w_r)$ (with $r = 1, 2, 3$), which represent the overlapping configuration, are given in terms of the Mandelstam map¹⁷⁾ from the complex plane with coordinate z into the (pants-shaped) ρ -plane (Fig. 1):

$$\rho(z) = \alpha_1 \log(z - 1) + \alpha_2 \log z. \quad (2.12)$$

The mapping $h_r(w_r)$ is defined by combining a logarithmic function that maps the r -th disk

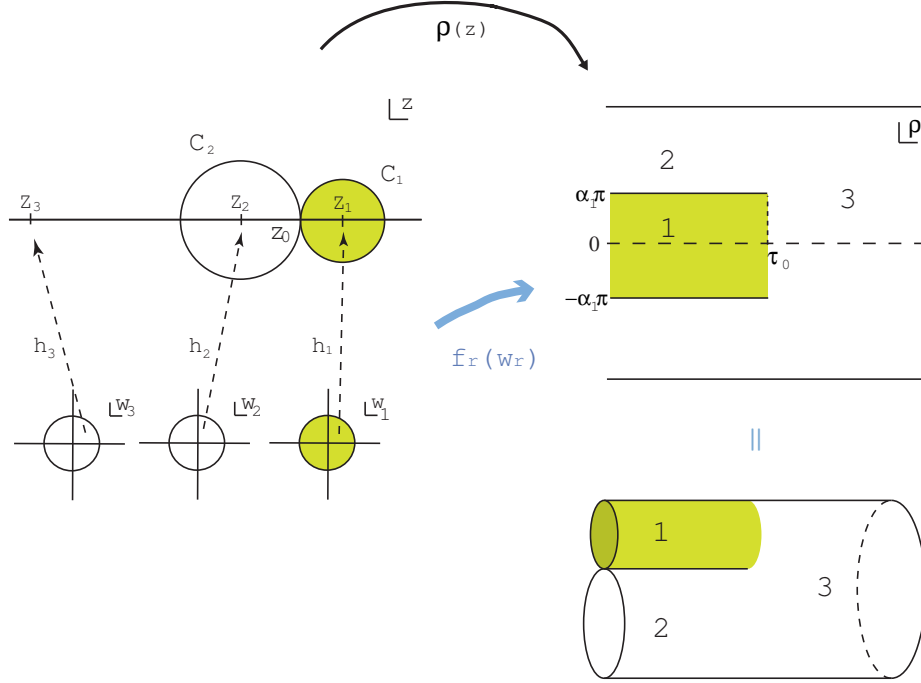


Fig. 1. $\rho(z)$ represents Mandelstam map. The images of string 1 and 2 at the interaction time τ_0 are the contours C_1 and C_2 , respectively. $C_1 + C_2$ becomes string 3.

to the r -th strip in the ρ -plane and the inverse of Mandelstam map:

$$h_r(w_r) = \rho^{-1}(f_r(w_r)), \quad f_r(w_r) = \alpha_r \log w_r + \tau_0 + i\beta_r. \quad (2.13)$$

In the logarithmic function $f_r(w_r)$, the argument of w_r should be taken from $-\pi$ to π . We divide it into two regions (corresponding to positive and negative argument) to define the parameter $\beta_r = \text{sgn}(\arg(w_r))\pi \sum_{s=1}^{r-1} \alpha_s$. The interaction time τ_0 is defined as $\tau_0 = \text{Re } \rho(z_0) = \sum_{r=1}^3 \alpha_r \log |\alpha_r|$ ($\alpha_3 = -\alpha_1 - \alpha_2$), where $z_0 = -\alpha_2/\alpha_3$ is the solution to $d\rho(z)/dz = 0$. The cyclic symmetry (2.9) can be demonstrated by the $SL(2, R)$ transformation in the z -plane. The 3-string vertex $\langle \hat{V}(1, 2, 3) |$ is given by

$$(\Phi_1(\alpha_1) \star \Phi_2(\alpha_2)) \cdot \Phi_3(\alpha_3) = \langle \hat{V}(1, 2, 3) | b_0^{-(3)} | \Phi_1(\alpha_1) \rangle_1 | \Phi_2(\alpha_2) \rangle_2 | \Phi_3(\alpha_3) \rangle_3. \quad (2.14)$$

The inserted operator $b_0^{-(3)}$ here comes from the dot product. From Eq. (2·8), this vertex can be expressed in terms of the LPP vertex (2·1) and some factors as

$$\begin{aligned} \langle \hat{V}(1, 2, 3) | b_0^{-(3)} = & - \int \frac{d\alpha_1}{2\pi} \frac{d\alpha_2}{2\pi} \frac{d\alpha_3}{2\pi} {}_1\langle \alpha_1 | {}_2\langle \alpha_2 | {}_3\langle \alpha_3 | {}_{123}\langle v_3 | \\ & \times b_0^{-(1)} \wp^{(1)} b_0^{-(2)} \wp^{(2)} b_0^{-(3)} \wp^{(3)}. \end{aligned} \quad (2\cdot15)$$

The basic properties of the \star product are summarized as

$$\Phi_1 \star \Phi_2 = -(-1)^{|\Phi_1||\Phi_2|} \Phi_2 \star \Phi_1, \quad (2\cdot16)$$

$$\begin{aligned} \Phi_1 \star (\Phi_2 \star \Phi_3) + (-1)^{|\Phi_1|(|\Phi_2|+|\Phi_3|)} \Phi_2 \star (\Phi_3 \star \Phi_1) \\ + (-1)^{|\Phi_3|(|\Phi_1|+|\Phi_2|)} \Phi_3 \star (\Phi_1 \star \Phi_2) = 0. \end{aligned} \quad (2\cdot17)$$

The first line [“(anti-)commutativity”] follows from Eq. (2·8) and the $SL(2, C)$ -invariance of the conformal vacuum. The second line [the “Jacobi identity”] is much more nontrivial and we need to use the critical dimension $d = 26$ to prove it.^{11), 21)}⁴ We note that this identity should not be confused with associativity, which generally does not hold for the closed string star product.

2.2. Action

In this subsection, we give a brief review of the HIKKO closed string field theory. We note, however, that the definition of the star product alone is sufficient to understand our claims. For this reason, the reader may skip to the next section to see our basic claims. We use the HIKKO construction to explain the ghost zero-mode convention of the physical state and to investigate an analogy with VSFT.

The action of the HIKKO closed string field theory is similar to that of Witten’s open string field theory,

$$S = \frac{1}{2} \Phi \cdot Q_B \Phi + \frac{1}{3} g \Phi \cdot (\Phi \star \Phi), \quad (2\cdot18)$$

where $Q_B = \oint j_B + \oint \tilde{j}_B$ is the conventional nilpotent BRST operator for a closed string:

$$Q_B = c_0^+(L_0 + \tilde{L}_0 - 2) + \frac{1}{2} c_0^-(L_0 - \tilde{L}_0) \quad (2\cdot19)$$

$$\begin{aligned} & + (M + \tilde{M}) b_0^+ + 2(M - \tilde{M}) b_0^- + Q'_B, \\ M = & - \sum_{n=1}^{\infty} n c_{-n} c_n, \quad \tilde{M} = - \sum_{n=1}^{\infty} n \tilde{c}_{-n} \tilde{c}_n, \end{aligned} \quad (2\cdot20)$$

⁴ Originally, this identity was shown using the Cremmer-Gervais identity¹⁸⁾ in the oscillator formulation. To prove this identity in terms of LPP language, the generalized gluing and re-smoothing theorem (GGRT) is essential.^{22)–24)}

$$Q'_B = \sum_{n \neq 0} \left(c_{-n} L_n^{\text{matter}} + \tilde{c}_{-n} \tilde{L}_n^{\text{matter}} \right) + \sum_{m, n, m+n \neq 0} \frac{m-n}{2} \left(c_m c_n b_{-m-n} + \tilde{c}_m \tilde{c}_n \tilde{b}_{-m-n} \right). \quad (2.21)$$

We have written the dependence on ghost zero modes explicitly for later convenience. The operator Q_B has the following properties with respect to the dot product and the \star product. For the dot product, we have

$$(Q_B \Phi_1(\alpha_1)) \cdot \Phi_2(\alpha_2) = -(-1)^{|\Phi_1|} \Phi_1(\alpha_1) \cdot (Q_B \Phi_2(\alpha_2)) - \pi \delta(\alpha_1 + \alpha_2) \langle I[\Phi_1](L_0 - \tilde{L}_0) \Phi_2 \rangle \quad (2.22)$$

from Eq. (2.4), using contour deformation in the CFT correlator and the relation $\{b_0^-, Q_B\} = \frac{1}{2}(L_0 - \tilde{L}_0)$. In particular, if Φ_1 or Φ_2 satisfies the level matching condition, the second term on the right-hand side vanishes, and therefore the “partial integration formula” holds. We can show that the BRST charge Q_B is a derivation with respect to the \star product:

$$Q_B(\Phi_1 \star \Phi_2) = (Q_B \Phi_1) \star \Phi_2 + (-1)^{|\Phi_1|} \Phi_1 \star (Q_B \Phi_2). \quad (2.23)$$

Here we have used Eq. (2.22), $(L_0 - \tilde{L}_0)|\Phi_1 \star \Phi_2\rangle = 0$, the contour deformation in Eq. (2.8) and anti-commutativity (i.e., the relation $\{Q_B, b_0^-\} = 0$).

In the action Eq. (2.18), the string fields are subject to some constraints: (1) the string field $|\Phi\rangle$ should have ghost number 3 (i.e., each \mathcal{O}_A in the expansion Eq. (2.3) is a ghost number 3 operator); (2) $|\Phi\rangle$ has odd Grassmann parity; (3) we impose the reality condition in the sense that we have

$$(|\Phi\rangle_2)^\dagger = \langle \hat{R}(1, 2) | \Phi \rangle_1; \quad (2.24)$$

(4) the level matching condition $(L_0 - \tilde{L}_0)|\Phi\rangle = 0$ (or $\wp|\Phi\rangle = |\Phi\rangle$) is imposed.

The action Eq. (2.18) is invariant under nonlinear gauge transformations of the string field. Thus we have

$$\delta_A \Phi = Q_B \Lambda + g(\Phi \star \Lambda - \Lambda \star \Phi), \quad (2.25)$$

where $|\Lambda\rangle$ is a gauge parameter that has ghost number 2 and even Grassmann parity and satisfies the level matching condition. We can easily confirm the gauge invariance $\delta_A S = 0$ using the nilpotency $Q_B^2 = 0$ and the properties of dot and star products (2.4), (2.8), (2.16), (2.17).

By expanding the string field Φ with respect to the ghost zero mode,

$$|\Phi\rangle = c_0^- |\phi\rangle + c_0^- c_0^+ |\psi\rangle + |\chi\rangle + c_0^+ |\eta\rangle, \quad (2.26)$$

the kinetic term of the action (2.18) becomes

$$\langle I[\Phi] b_0^- Q_B \Phi \rangle = \langle I[\phi] (L_0 + \tilde{L}_0 - 2) \phi \rangle + \cdots, \quad (2.27)$$

where we have omitted the α -dependent factor. This shows that the physical sector is contained in the slot $c_0^- |\phi\rangle$ in the expansion Eq. (2.26) in our convention. In fact, the gauge in which $\psi = \chi = \eta = 0$ is adopted in Ref. 11) in order to obtain the gauge fixed action from the gauge invariant action (2.18). By contrast, we remove the (internal) ghost number constraint from $c_0^- |\phi\rangle$ to include FP ghosts in SFT.

§3. A universal nonlinear relation for the boundary state

3.1. Summary of our previous results

In our previous paper,¹⁰⁾ we use the boundary state of the Dp -brane in a flat background with a constant electro-magnetic flux $F_{\mu\nu}$.⁵

$$|B(x^\perp)\rangle = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \mathcal{O} \tilde{\alpha}_{-n} + \sum_{n=1}^{\infty} (c_{-n} \tilde{b}_{-n} + \tilde{c}_{-n} b_{-n}) \right) \quad (3.1)$$

$$\times c_0^+ c_1 \tilde{c}_1 |p^\parallel = 0, x^\perp\rangle \otimes |0\rangle_{gh},$$

$$\mathcal{O}^\mu_\nu = [(1+F)^{-1}(1-F)]^\mu_\nu, \quad \mu, \nu = 0, 1, \dots, p, \quad (3.2)$$

$$\mathcal{O}^i_j = -\delta^i_j, \quad i, j = p+1, \dots, d-1. \quad (3.3)$$

Here, p^\parallel (resp., x^\perp) is the momentum (resp., coordinate) along the Neumann (resp., Dirichlet) directions. The ghost sector is fixed by the boundary conditions $(c_n + \tilde{c}_{-n})|B\rangle_{gh} = (b_n - \tilde{b}_{-n})|B\rangle_{gh} = 0$. The state $|0\rangle_{gh}$ is the $SL(2, C)$ invariant vacuum. For the matter sector, the relation $(L_n^{\text{matter}} - \tilde{L}_{-n}^{\text{matter}})|B(x^\perp)\rangle = 0$ is satisfied, because \mathcal{O} in the exponent is an orthogonal matrix. This implies $Q_B|B(x^\perp)\rangle = 0$ for the conventional BRST operator Q_B (2.19).

We need to slightly modify the boundary state to follow the convention of the string fields in the previous section.¹⁰⁾ Here, we define

$$|\Phi_B(x^\perp, \alpha)\rangle \equiv c_0^- b_0^+ |B(x^\perp)\rangle \otimes |\alpha\rangle. \quad (3.4)$$

We include the α parameter here to define the \star product and the factor $c_0^- b_0^+$. The ghost factor $c_0^- b_0^+$ has the effect of replacing c_0^+ with c_0^- , so that the state is placed in the correct

⁵ We note that the notation for the ghost fields used here is slightly different from that in our previous paper.¹⁰⁾ The correspondence between them is explained in Appendix A.

slot of the string field (2.26). The first nontrivial statement in Ref. 10) is that this modified boundary state satisfies the idempotency relation for $\alpha_1 \alpha_2 > 0$,

$$\Phi_B(x^\perp, \alpha_1) \star \Phi_B(y^\perp, \alpha_2) = \delta^{d-p-1}(x^\perp - y^\perp) \mathcal{C} c_0^+ \Phi_B(y^\perp, \alpha_1 + \alpha_2). \quad (3.5)$$

The constant factor \mathcal{C} is given formally in terms of the determinant of an infinite-dimensional matrix and was not fixed analytically in Ref. 10). It turns out that it can be determined by using the so-called Cremmer-Gervais identity,¹⁸⁾ which is explained in Appendix B. It takes a very simple form only for the critical dimension, $d = 26$, in which case we have

$$\mathcal{C} = K^3 |\alpha_1 \alpha_2 (\alpha_1 + \alpha_2)|, \quad (3.6)$$

where K is an infinite constant that depends on the cutoff. It is universal in the sense that it is independent of $F_{\mu\nu}$, x^\perp and p . Therefore, if we change the normalization of Φ_B so that $\tilde{\Phi}(x^\perp, \alpha) \equiv \frac{-1}{g_o |\alpha|} \Phi_B(x^\perp, \alpha)$, together with the inclusion of the usual open string coupling ($g_o = g^{1/2}$), the dependence on the parameter α formally drops from the idempotency relation,

$$\delta^{d-p-1}(x^\perp - y^\perp) \mathcal{Q} \tilde{\Phi}(x^\perp, \alpha_1 + \alpha_2) + g' \tilde{\Phi}(x^\perp, \alpha_1) \star \tilde{\Phi}(y^\perp, \alpha_2) = 0. \quad (3.7)$$

Here, $\mathcal{Q} \equiv \hat{\alpha}^2 c_0^+$ is the “pure ghost BRST operator,” which we discuss below, and $g' = g_o K^{-3}$ is the renormalized string coupling constant.

If we wish to interpret (3.7) as an equation of motion of a possible vacuum theory, the appearance of the delta function is annoying, as it depends on the number of the transverse directions explicitly. It can be removed, however, if we take a superposition of Dp -brane boundary states along the transverse direction,

$$\Phi_f(\alpha) \equiv \int d^{d-p-1} x^\perp f(x^\perp) \tilde{\Phi}(x^\perp, \alpha). \quad (3.8)$$

Suppose f satisfies the equation

$$f^2(x^\perp) = f(x^\perp), \quad (3.9)$$

Then, the relation for Φ_f can be written in the universal form

$$\mathcal{Q} \Phi_f(\alpha_1 + \alpha_2) + g' \Phi_f(\alpha_1) \star \Phi_f(\alpha_2) = 0. \quad (3.10)$$

We remark that the constraint (3.9) has the form of a “noncommutative” soliton for the commutative ring of functions in the transverse direction.⁶

⁶ The algebraic constraint for f is satisfied by $f(x^\perp) = 1$ (if $x^\perp \in \Sigma$) and $f(x^\perp) = 0$ (otherwise) for some subset Σ of \mathbf{R}^{d-p-1} . This fixing of f describes the distribution of D-branes in the transverse direction. The discrete nature of f is regularized in the noncommutative space-time, namely by including a B field along the transverse direction.²⁵⁾ In this case, the equation for f should be changed to $f \star f = f$, where \star is a product that reflects the noncommutativity. This is similar to the equation of the noncommutative soliton.⁹⁾ We hope to come back to this issue in a future paper.

The second nontrivial observation made in Ref. 10) is the deformation of (3.5). We considered deformations of the type

$$\delta\Phi_B(x^\perp, \alpha) = \oint \frac{d\sigma}{2\pi} V(\sigma) \Phi_B(x^\perp, \alpha). \quad (3.11)$$

This describes infinitesimal deformations of the boundary condition. It can be physically interpreted as describing the infinitesimally curved D-brane induced by the collective modes of the open strings. In order that the variation preserves the idempotency relation (3.5), we need to impose the condition

$$\begin{aligned} \delta\Phi_B(x^\perp, \alpha_1) \star \Phi_B(y^\perp, \alpha_2) + \Phi_B(x^\perp, \alpha_1) \star \delta\Phi_B(y^\perp, \alpha_2) \\ = \delta^{d-p-1}(x^\perp - y^\perp) \mathcal{C} c_0^+ \delta\Phi_B(x^\perp, \alpha_1 + \alpha_2). \end{aligned} \quad (3.12)$$

We examined the two simplest cases, in which $V(\sigma)$ is given by (1) the scalar-type deformation $V_S(\sigma) =: e^{ik_\mu X^\mu(\sigma)}$: and (2) the vector-type deformation $V_V(\sigma) =: \zeta_\nu \partial_\sigma X^\nu e^{ik_\mu X^\mu(\sigma)}$:. We have proved that the relation (3.12) requires the on-shell conditions for open string tachyon and vector particle,

$$k_\mu G^{\mu\nu} k_\nu = \frac{1}{\alpha'} \quad \text{for } V = V_S, \quad k_\mu G^{\mu\nu} k_\nu = 0 \quad \text{for } V = V_V, \quad (3.13)$$

where

$$G^{\mu\nu} \equiv [(1 + F)^{-1} \eta (1 - F)]^{\mu\nu}$$

is the “open string metric”.²⁵⁾

We also demonstrated that the vector-type variation has a “gauge symmetry” of the form $\zeta_\nu = \zeta_\nu + \epsilon k_\nu$, owing to the relation

$$\oint \frac{d\sigma}{2\pi} (k_\nu \partial_\sigma X^\nu) e^{ik_\mu X^\mu(\sigma)} \Phi_B = -i \oint \frac{d\sigma}{2\pi} \partial_\sigma (e^{ik_\mu X^\mu(\sigma)}) \Phi_B = 0. \quad (3.14)$$

On the other hand, the requirement of the transversality condition

$$\zeta \cdot k \equiv \zeta_\mu G^{\mu\nu} k_\nu = 0 \quad (3.15)$$

is rather subtle, because we encounter a coefficient of the form $0 \times \infty$ multiplying this factor. While the appearance of such a subtlety is inevitable in the operator formalism, we prove in §6 that a regularized expression is obtained in the CFT approach and that the transversality condition is indeed needed.

To summarize, in all the examples we studied, the deformation of the idempotency relation precisely reproduces the spectrum of an open string. This convincingly shows that

Eq. (3.5) is a very good characterization of D-branes, including the infinitesimally curved branes.

We now give an intuitive proof of the idempotency relation Eq. (3.5). We first explain the nature of the boundary state as a surface state. Consider an inner product between a boundary state $|B\rangle$ and a vector in the closed string Fock space $|\phi\rangle = \mathcal{O}|0\rangle$. As is well-known, $\langle B|\phi\rangle$ gives a one point function on a disk $\langle \mathcal{O}(0)\rangle$ with the boundary condition at the boundary $|w| = 1$ specified by the boundary state. By the conformal mapping $w = e^{\tau+i\sigma}$, the disk is mapped to a half-infinite cylinder that is cut at $\tau = 0$. Therefore, as depicted in the first figure in Fig. 2, the boundary state is characterized by two operations as a surface state: (1) to cut the infinite cylinder at $\tau = 0$, and (2) to set an appropriate boundary condition at the edge.

When we calculate the star product between the boundary states $(\langle B| \star \langle B|)|\phi\rangle$, we prepare a pants diagram which represents the HIKKO vertex (the second figure of Fig. 2), and we attach boundary states at its two legs. As the surface states, they have the effect of stripping off the two legs at the interaction time τ_0 and set the same boundary conditions along the two circles. Suppose we can ignore the curvature singularity at the interaction point. Then we are left with a half infinite cylinder with the boundary condition of $|B\rangle$. This is in effect the same as taking the inner product $\langle B|\phi\rangle$ (Fig. 2).

$$\begin{array}{ccc}
\begin{array}{c} \tau = \infty \qquad \tau = 0 \\ \langle B| \quad \quad \quad |\phi\rangle \\ \hline \langle B|\phi \rangle \end{array} & = & \begin{array}{c} \tau = 0 \\ \text{Cylinder} \\ \langle \mathcal{O}(0) \rangle \end{array} \\
\\
\begin{array}{c} \text{Pants diagram with } \langle B| \text{ legs and } |\phi\rangle \text{ boundary} \\ \langle V_3[B]|B\rangle|\phi\rangle = (\langle B| \star \langle B|)|\phi\rangle \end{array} & = & \begin{array}{c} \tau = 0 \\ \text{Cylinder} \\ \langle B|\phi \rangle \end{array}
\end{array}$$

Fig. 2. Star product of boundary states.

From the above discussion, we see the uniqueness of the boundary state with respect to the star product of the closed string. From the nontrivial topology of the pants diagram, it is difficult to imagine that anything other than the boundary state can satisfy the idempotency relation. In this sense, although it is difficult to prove rigorously, the solution set of the idempotency relation Eq. (3.5) seems to be identical to the set of consistent boundary states.

3.2. Vacuum theory for closed string fields

Since the idempotency relation (3.10) for the boundary states takes a universal form, it may be natural to consider the action whose equation of motion is given by this relation. We now describe some properties of such a theory in order to find an analogy to VSFT.⁴⁾ This theory is very similar to the HIKKO theory, which we reviewed in the previous section. Consider an action of the form

$$S = \frac{1}{2} \Phi \cdot \mathcal{Q}\Phi + \frac{g'}{3} \Phi \cdot (\Phi \star \Phi), \quad (3.16)$$

where the operator $\mathcal{Q} = \hat{\alpha}^2 c_0^+$ is that appearing in Eq. (3.7). It has ghost number 1 and satisfies the same type of relations as the original BRST operator Q_B ,

$$\begin{aligned} \mathcal{Q}^2 &= 0, & (\mathcal{Q}\Phi_1) \cdot \Phi_2 &= -(-1)^{|\Phi_1|} \Phi_1 \cdot \mathcal{Q}\Phi_2, \\ \mathcal{Q}(\Phi_1 \star \Phi_2) &= (\mathcal{Q}\Phi_1) \star \Phi_2 + (-1)^{|\Phi_1|} \Phi_1 \star (\mathcal{Q}\Phi_2). \end{aligned} \quad (3.17)$$

The first equation here is obvious. The last two identities can be proved using the relation $c_0^+ = \frac{1}{2} \oint \frac{dw}{2\pi i} w^{-2} c(w) + \frac{1}{2} \oint \frac{d\bar{w}}{2\pi i} \bar{w}^{-2} \tilde{c}(\bar{w})$ and Eqs. (2.4) and (2.8). Note that $\hat{\alpha}^2$ is necessary to cancel the conformal factor.⁷ They ensure the gauge invariance of the action under the transformation

$$\delta_A \Phi = \mathcal{Q}\Lambda + g'(\Phi \star \Lambda - \Lambda \star \Phi). \quad (3.18)$$

The equation of motion for the action (3.16) is

$$b_0^- (\mathcal{Q}\Phi + g'(\Phi \star \Phi)) = 0. \quad (3.19)$$

This is very similar to the relation (3.10). Suppose that it can be extended to the case $\alpha_1 \alpha_2 < 0$.⁸ Formally, its solution is given by

$$\Phi_0(\tilde{\alpha}) = \lim_{M \rightarrow \infty} \frac{1}{2M} \int_{-M}^M d\alpha e^{i\alpha \tilde{\alpha}} \Phi_f(\alpha). \quad (3.20)$$

We note that the appearance of the Fourier transformation with respect to the parameter α is typical in the HIKKO computation of string amplitudes,²⁶⁾ where we must impose the condition that all the “momentum” in the external lines have the same $\tilde{\alpha}$. The divergent normalization factor $(2M)^{-1}$ can be removed through the renormalization of the coupling constant: $g' \rightarrow \tilde{g} = g'(2M)^{-1}$. This situation is also parallel to that in Ref. 26).

⁷ The same operator, $\hat{\alpha}^2 c_0^+$, was also considered in Ref. 21) in a different context. We can also check these identities in the oscillator language using properties of the Neumann coefficients.

⁸ That such an extension would be possible seems natural from the cyclicity of the product (2.9). There exist, however, some subtleties similar to the divergence of the norm of the boundary state.

We now consider the re-expansion of the action (3.16) around the nontrivial classical solution $\Phi_0(\tilde{\alpha})$,

$$S' = \frac{1}{2}\Phi \cdot \mathcal{Q}_0\Phi + \frac{\tilde{g}}{3}\Phi \cdot (\Phi \star \Phi) + S_0, \quad (3.21)$$

where the new kinetic term \mathcal{Q}_0 is given by

$$\mathcal{Q}_0\Phi = \mathcal{Q}\Phi + \tilde{g}(\Phi_0(\tilde{\alpha}) \star \Phi - (-1)^{|\Phi|}\Phi \star \Phi_0(\tilde{\alpha})), \quad (3.22)$$

and $S_0 = -\frac{\tilde{g}}{6}\Phi_0 \cdot (\Phi_0 \star \Phi_0)$ corresponds to “potential height.” This new kinetic term of S' given in Eq. (3.21) also satisfies the three identities (3.17), and therefore it possesses gauge invariance.

The above consideration suggests that we can consider the “vacuum version” of closed string field theory as an analogue of the vacuum string field theory (VSFT) of the Witten-type *open* string field theory.⁴⁾ As in VSFT, this pure ghost BRST operator has no physical states. As we have observed in Ref. 10) and will refine the results in the following, we might interpret that the boundary state is a classical solution Φ_0 to the equation of motion (3.19) with $\mathcal{Q} = \hat{\alpha}^2 c_0^+$ and \mathcal{Q}_0 given in Eq. (3.22) has *open* string spectrum on the D-brane. This is strikingly similar to VSFT scenario.⁴⁾

We note that the string coupling \tilde{g} is equal to the open string coupling $g_{\text{open}} = g_{\text{closed}}^{1/2}$ up to an infinite constant factor. This may be related to the fact that there seems to be no physical closed string sector in \mathcal{Q}_0 , as far as we studied. In this sense, the vacuum theory that we are considering may be regarded as a purely open string field theory, while we are using the closed string variables and thus the properties of the star product are very different.

Finally, we comment that there may be possibilities other than considering the vacuum theory. We note that there is a close analogy between Eq. (3.10) and the wedge state algebra⁶⁾ in open string field theory,

$$W_n \star W_m = W_{n+m-1}, \quad W_n = (|0\rangle)_\star^{n-1}, \quad (3.23)$$

with the correspondence $\alpha \leftrightarrow (n-1)$. This link becomes more precise if we take the large n limit, where we can regard the parameter n (after rescaling) as a continuous parameter. In this limit, the wedge state becomes the sliver state. We hope to come back to this analogy in the future to elucidate the explicit link between our closed string formulation and VSFT.

§4. Comments on the Witten-type vertex in nonpolynomial closed SFT

Here we briefly turn our attention to another formulation that has been well-examined, the non-polynomial closed string field theory,^{13)–16)} and we discuss the idempotency equation of boundary states in this context.

The non-polynomial closed SFT was constructed as a direct extension of Witten's open string field theory to closed strings. It has the merit that it does not contain extra parameters. On the other hand, it is known that an infinite number of higher-order interaction terms are necessary in the classical action in order for the theory to cover the moduli space properly.^{14), 15)} While a formal method to construct all vertices is known in terms of the LPP formulation,¹⁶⁾ it seems that it is impractical to perform explicit computations to all orders.

For this reason, we restrict ourselves to a 3-string vertex, where a computation similar to that given in Ref. 10) is possible with the knowledge of Neumann coefficients of Witten's open SFT.²⁷⁾ The 3-string vertex $\langle V_W(1, 2, 3) |$ of a nonpolynomial closed SFT is defined using the LPP vertex as

$$\langle V_W(1, 2, 3) | = \langle v_{3W} | b_0^{-(1)} \wp^{(1)} b_0^{-(2)} \wp^{(2)} b_0^{-(3)} \wp^{(3)}, \quad (4.1)$$

where $\langle v_{3W} |$ is the LPP 3-string vertex (2.1) with conformal mappings $h_r(w_r) = h^{-1} \left(e^{\frac{2\pi r}{3} i} h(w_r)^{\frac{2}{3}} \right)$ (where $r = 1, 2, 3$ and $h(w) := \frac{1+iw}{1-iw}$), which realize the Witten-type overlapping. These maps are identical to those appearing in the open string. This implies that the Neumann coefficients of the (anti-) holomorphic part of $\langle v_{3W} |$ coincide with those of Witten's open SFT. We carried out the calculation of the star product of the boundary states by replacing the Neumann coefficients in a previous paper¹⁰⁾ with those of the Witten theory. The nonlinear relations for the Neumann coefficients essential in the computation are actually the same:

$$\sum_{t,l} V_{nl}^{rt} V_{lm}^{ts} = \delta_{nm} \delta^{rs}, \quad \sum_{t,l} V_{nl}^{rt} V_{l0}^{ts} = V_{n0}^{rs}, \quad \sum_{t,l} V_{0l}^{rt} V_{l0}^{ts} = 2V_{00}^{rs}, \quad (4.2)$$

$$\sum_{t,l} X_{nl}^{rt} X_{lm}^{ts} = \delta_{nm} \delta^{rs}, \quad \sum_{t,l} X_{nl}^{rt} X_{l0}^{ts} = X_{n0}^{rs}. \quad (4.3)$$

Therefore, the computation of the “ \star -product” of the boundary states is parallel to that in Ref. 10).⁹ The result is

$$\begin{aligned} \Phi_B(x^\perp) \star \Phi_B(y^\perp) &= \det^{-\frac{d}{2}}((1 - (V^{33})^2)) \det(1 - (X^{33})^2) \mathbf{c} \\ &\times \delta^{d-p-1}(x^\perp - y^\perp) c_0^+ b_0^- \Phi_B(x^\perp), \end{aligned} \quad (4.4)$$

where we have defined $|\Phi_B(x^\perp)\rangle = c_0^- b_0^+ |B(x^\perp)\rangle$, which is the same form as Eq. (3.4), while

⁹ Here, we have used the notation in Ref. 28) for Neumann coefficients. For the matter sector, the relations (4.2) have the same form as those of the light-cone gauge SFT.²⁹⁾ There is a difference, however. In the present case, the rank of the matrices $(1 - n^{(m)2}), (1 - n^{(g)2})$ is half of their size, where we define $n^{(m)} = \begin{pmatrix} -V^{11} & -V^{12} \\ -V^{21} & -V^{22} \end{pmatrix}, n^{(g)} = \begin{pmatrix} -X^{11} & -X^{12} \\ -X^{21} & -X^{22} \end{pmatrix}$. This causes extra divergent factor \mathbf{c} .

it does not contain the α -sector.¹⁰ Up to a constant prefactor, this nonlinear equation has the same form as the HIKKO-type vertex (3.5).

In this case, however, the prefactor \mathbf{c} in the above equation is strongly divergent. Specifically, there appears a factor of the form $\prod_{\sigma \in \text{overlap}} \delta(0)$, which should be regularized somehow. This divergence results from the geometrical nature of the Witten-type vertex (Fig. 3). The problem is that it attaches two boundary states point-wise for half of the boundary. Roughly speaking, the boundary state contains a factor of the form $\prod_{\sigma} \delta(X(\sigma) - \tilde{X}(\sigma))$, because it identifies the left and right movers. On the other hand, the vertex operator contains the factor $\prod_{\sigma, \sigma'} \delta(X^{(1)}(\sigma) - X^{(2)}(\sigma')) \delta(\tilde{X}^{(1)}(\sigma) - \tilde{X}^{(2)}(\sigma'))$, where σ and σ' are the coordinates of the attached points on each string. If we take the star product of two boundary states, we are left with $\delta(0)$ for each point σ where two boundary states are attached.

In the HIKKO \star product (in the case $\alpha_1 \alpha_2 > 0$), no such divergent factor exists, because the overlapping part of strings 1 and 2 is only a point, i.e. the interaction point (z_0 in Fig. 1). This makes the HIKKO vertex appropriate for our purpose.

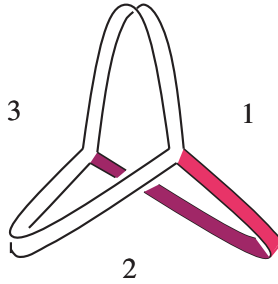


Fig. 3. The halves of strings 1 and 2 overlap each other in the “ \star -product.”

§5. Proof of idempotency as a background-independent relation

In this section, we give a proof of the idempotency relation of boundary states using the CFT technique described in the preceding sections. Compared with the proof carried out in Ref. 10) in terms of oscillators, the argument in this section has the merit that it does not depend on the particular background of the closed string. It also illuminates the (world-sheet) geometrical nature of the relation.

We first recall the basic constraints of the consistent boundary state in a general background. First, it must be invariant under the conformal transformation at the boundary:

¹⁰ We define the “ \star -product” by $\langle V_W(1, 2, 3) | \Phi_1 \rangle_1 | \Phi_2 \rangle_2 = {}_3 \langle \Phi_1 \star \Phi_2 |$. The extra b_0^- factor on the right-hand side comes from the difference between the conventions of the HIKKO (2.15) and the non-polynomial formulation (4.1) in the three string vertex.

$$\mathcal{L}_n|B\rangle = 0, \quad \mathcal{L}_n \equiv L_n - \tilde{L}_{-n}. \quad (5.1)$$

This linear condition alone, however, is not sufficient. In order to have a well-defined Hilbert space in the open string channel, we need to impose the Cardy condition;³⁰⁾ that is, for two arbitrary boundary states $|B\rangle$ and $|B'\rangle$, we calculate the inner product with a propagator and apply the modular transformation,

$$\langle B|q^{\frac{1}{2}(L_0+\tilde{L}_0)}|B'\rangle = \chi_{BB'}(q) = \sum_i N_{BB'}^i \chi_i(\tilde{q}). \quad (5.2)$$

Here, the index i in the summation labels the irreducible representations, and $\chi_i(\tilde{q})$ is the corresponding character. The Cardy condition is that the coefficient $N_{BB'}^i$ must be a non-negative integer for any i . In the following, we refer to the first requirement (5.1) as the “weak condition” and the second (5.2) as the “strong condition.”

We conjecture that the *strong* condition is equivalent to our idempotency relation, since both are quadratic with respect to the string fields. We have confirmed this explicitly for flat backgrounds, including toroidal compactification, as we briefly explain in the discussion section. However, this is possible only for cases in which we have an explicit oscillator representation, and a background-independent proof is yet to be completed. Therefore, we present only the proof of the weak condition in this section. The assertion that we demonstrate is

$$\mathcal{L}_n|B_a\rangle = 0 \quad (a = 1, 2) \quad \rightarrow \quad \mathcal{L}_n|B_1 \star B_2\rangle = 0 \quad (5.3)$$

in the LPP formulation. The last expression is equivalent to

$$(\mathcal{L}_n\Phi) \cdot (\Phi_{B_1} \star \Phi_{B_2}) = \langle h_3[\wp\mathcal{L}_n\Phi]h_1[\mathcal{O}_1]h_2[\mathcal{O}_2]\rangle = 0, \quad (\text{for any } \Phi) \quad (5.4)$$

where \mathcal{O}_r is the operator that corresponds to the boundary state $|B_r\rangle = \mathcal{O}_r|0\rangle$, up to the ghost sector and the α conservation factor.

In the following, we present the derivation in which L_n and \tilde{L}_n are restricted to the matter sector. This restriction simplifies the computation and helps to illuminate the essence of our proof. The ghost sector is universal for any background, and in a previous paper¹⁰⁾ we gave a proof in the operator formulation for the case of a flat background. For this reason, ignoring the ghost sector will not be so problematic.

The Virasoro generators L_n are originally defined as the coefficients of the stress energy tensor in w_r coordinates, $T(w_r) = \sum_n L_n w_r^{-n-2}$. For our purpose, it is more convenient to use the coordinates $\zeta_r = \log(w_r)$, because in this case, the connection conditions between

three patches at the vertex become the simplest. The conformal transformation of the stress-energy tensor is the standard one,

$$T(\zeta) = \left(\frac{dw}{d\zeta}\right)^2 T(w) + \frac{c}{12} S(w, \zeta), \quad S(z, w) = \frac{d^3 z}{dw^3} - \frac{3}{2} \left(\frac{d^2 z}{dw^2}\right)^2. \quad (5.5)$$

In our case, the stress energy tensor is written $T(\zeta) = \sum_n L_n e^{-n\zeta} - \frac{c}{24}$.

At the boundary of the disk $|w| = 1$ [or $\text{Re}(\zeta) = 0$], the operators \mathcal{L}_n appear in the combination

$$\mathcal{T}(\sigma) = \sum_n \mathcal{L}_n e^{-in\sigma} = T(i\sigma) - \tilde{T}(-i\sigma). \quad (5.6)$$

This operator describes the reparametrization at the boundary. A (boundary) primary field of dimension Δ has the following relation with \mathcal{L}_n :

$$\mathcal{L}_n V(\sigma)|B\rangle = e^{in\sigma} \left(-i\frac{d}{d\sigma} + n\Delta\right) V(\sigma)|B\rangle. \quad (5.7)$$

This corresponds to the reparametrization $\delta\sigma = -ie^{in\sigma}$. As a special example, the operator $\mathcal{T}(\sigma)$ itself transforms under the boundary reparametrization according to

$$\mathcal{T}(\sigma) = \mathcal{T}(\sigma') \left(\frac{d\sigma'}{d\sigma}\right)^2. \quad (5.8)$$

There is no anomaly term here, as there is no central extension for \mathcal{L}_n .

In the following, we replace the expression (5.4) by applying the overall conformal transformation $\rho(z)$:

$$\langle f_3[\wp \mathcal{L}_n \Phi] f_1[\mathcal{O}_1] f_2[\mathcal{O}_2] \rangle = 0, \quad (\text{for any } \Phi) \quad (5.9)$$

where

$$f_r(w_r) = \rho(h_r(w_r)) = \alpha_r \log w_r + \tau_0 + i\beta_r = \alpha_r \zeta_r + \tau_0 + i\beta_r. \quad (5.10)$$

The correlation function on the left-hand side should be evaluated on Mandelstam diagram. The generalized gluing and resmoothing theorem (GGRT)^{22)–24)} ensures that this expression is proportional to the original one, Eq. (5.4).

With the above preparation, the proof of the assertion is straightforward. The strategy is to rewrite the operator \mathcal{L}_n in terms of the contour integration $\int_{-\pi}^{\pi} \frac{d\sigma_3}{2\pi} e^{in\sigma_3} \mathcal{T}_3(\sigma_3)$ and to use the connection condition at the boundary that is implied by the 3-string vertex. This makes these operators act on \mathcal{O}_r and annihilate them under the assumption. In this process, we need to change the coordinate σ_3 to σ_r ($r = 1, 2$). We write the corresponding boundary stress-energy tensors as $\mathcal{T}_r(\sigma_r)$. These are related as $\mathcal{T}_3(\sigma_3) = \mathcal{T}_r(\sigma_r)(\alpha_3/\alpha_r)^2$, from Eq. (5.10). We

also use the following additional notation:

$$\wp \mathcal{L}_n = \mathcal{L}_n \wp_n, \quad \wp_n \equiv \oint \frac{d\sigma}{2\pi} e^{i\sigma(\mathcal{L}_0 - n)}, \quad (5.11)$$

$$\Sigma_r(\sigma_r) \equiv f_3^{-1}(f_r(\sigma_r)) = \frac{1}{\alpha_3}(\alpha_r \sigma_r + \beta_r - \beta_3). \quad (5.12)$$

Then, the proof is given as follows:

$$\begin{aligned} & \langle f_3[\wp \mathcal{L}_n \Phi] f_1[\mathcal{O}_1] f_2[\mathcal{O}_2] \rangle = \langle f_3[\mathcal{L}_n \wp_n \Phi] f_1[\mathcal{O}_1] f_2[\mathcal{O}_2] \rangle \\ &= \int_{-\pi}^{\pi} \frac{d\sigma_3}{2\pi} e^{in\sigma_3} \langle f_3[\mathcal{T}_3(\sigma_3) \wp_n \Phi] f_1[\mathcal{O}_1] f_2[\mathcal{O}_2] \rangle \\ &= - \int_{-\pi}^{\pi} \frac{d\sigma_1}{2\pi} e^{in\Sigma_1(\sigma_1)} \left(\frac{d\Sigma_1}{d\sigma_1} \right)^{-1} \langle f_3[\wp_n \Phi] f_1[\mathcal{T}_1(\sigma_1) \mathcal{O}_1] f_2[\mathcal{O}_2] \rangle \\ &\quad - \int_0^{2\pi} \frac{d\sigma_2}{2\pi} e^{in\Sigma_2(\sigma_2)} \left(\frac{d\Sigma_2}{d\sigma_2} \right)^{-1} \langle f_3[\wp_n \Phi] f_1[\mathcal{O}_1] f_2[\mathcal{T}_2(\sigma_2) \mathcal{O}_2] \rangle = 0. \end{aligned} \quad (5.13)$$

We note that we have changed the integration range of σ_2 in the last expression. This is because the function Σ_2 has a jump at $\sigma_2 = 0$, while it is continuous at $\sigma_2 = \pm\pi$, if we require periodic boundary conditions for Σ_2 .

While our proof looks straightforward, there is a subtle point that must be treated carefully. This regards the singularity at the interaction point. In the final line of Eq. (5.13), there appear functions of the form $e^{in\Sigma_r(\sigma_r)}$. These functions are, as we previously mentioned, not continuous. While this does not seem problematic in the above calculation, we have to treat it more seriously in the argument given in the next section.

§6. Derivation of the open string spectrum

6.1. Fluctuations around the idempotency equation

In Ref. 10), we examined variations of the idempotency relation (3.12) and derived the on-shell conditions for the open string modes. The analysis was, however, restricted to lower excited states, namely the tachyon and the massless vector particle. Analysis for the higher excited modes is not feasible, because the computation becomes complicated. In the LPP approach, however, we can carry out a more systematic study. The idempotency equation (3.5) can be rewritten in LPP language as

$$\begin{aligned} & \langle h_1[b_0^- \Phi_B(x^\perp)] h_2[b_0^- \Phi_B(y^\perp)] h_3[b_0^- \wp \Phi] \rangle \\ &= -\delta^{d-p-1}(x^\perp - y^\perp) \mathcal{C} \langle I[c_0^+ \Phi_B(x^\perp)] b_0^- \Phi \rangle, \end{aligned} \quad (6.1)$$

where we have assigned $\alpha = -\alpha_1 - \alpha_2$ to the arbitrary Φ and dropped the factor of $2\pi\delta(0)$ in the α sector. The variation of the idempotency equation (3.12) for the deformation Eq. (3.11)

in the matter sector can be rewritten in the same way. As in the previous section, we need to apply the conformal transformations $\rho(z)$ to the left and f_3 to the right to use the coordinate system ζ . The GGRT implies that the relation (6.1) is unchanged as long as the total central charge vanishes, i.e., for the critical dimension. The left-hand side of Eq. (3.12) becomes (with $J \equiv f_3 \circ I$)

$$\begin{aligned}
& \langle f_1[b_0^- \delta \Phi_B(x^\perp)] f_2[b_0^- \Phi_B(y^\perp)] f_3[b_0^- \wp \Phi] \rangle \\
& + \langle f_1[b_0^- \Phi_B(x^\perp)] f_2[b_0^- \delta \Phi_B(y^\perp)] f_3[b_0^- \wp \Phi] \rangle \\
& = \oint \frac{d\sigma_1}{2\pi} \langle f_1[b_0^- V(\sigma_1) \Phi_B(x^\perp)] f_2[b_0^- \Phi_B(y^\perp)] f_3[b_0^- \wp \Phi] \rangle \\
& + \oint \frac{d\sigma_2}{2\pi} \langle f_1[b_0^- \Phi_B(x^\perp)] f_2[b_0^- V(\sigma_2) \Phi_B(y^\perp)] f_3[b_0^- \wp \Phi] \rangle \\
& = \left\langle f_1[b_0^- \Phi_B(x^\perp)] f_2[b_0^- \Phi_B(y^\perp)] f_3 \left[\left(\oint \frac{d\sigma_1}{2\pi} \Sigma_1[V(\sigma_1)] + \oint \frac{d\sigma_2}{2\pi} \Sigma_2[V(\sigma_2)] \right) b_0^- \wp \Phi \right] \right\rangle \\
& = -\delta^{d-p-1}(x^\perp - y^\perp) \mathcal{C} \\
& \quad \times \left\langle J[c_0^+ \Phi_B(x^\perp)] f_3 \left[b_0^- \left(\oint \frac{d\sigma_1}{2\pi} \Sigma_1[V(\sigma_1)] + \oint \frac{d\sigma_2}{2\pi} \Sigma_2[V(\sigma_2)] \right) \wp \Phi \right] \right\rangle \\
& = -\delta^{d-p-1}(x^\perp - y^\perp) \mathcal{C} \\
& \quad \times \left\langle J \left[c_0^+ b_0^- \wp \left(\oint \frac{d\sigma_1}{2\pi} \Sigma_1[V(\sigma_1)] + \oint \frac{d\sigma_2}{2\pi} \Sigma_2[V(\sigma_2)] \right) \Phi_B(x^\perp) \right] f_3[\Phi] \right\rangle, \tag{6.2}
\end{aligned}$$

where we have used Eq. (6.1). Similarly, the right-hand side of Eq. (3.12) becomes

$$\begin{aligned}
& -\delta^{d-p-1}(x^\perp - y^\perp) \mathcal{C} \langle J[c_0^+ \delta \Phi_B(x^\perp)] f_3[b_0^- \Phi] \rangle \\
& = \delta^{d-p-1}(x^\perp - y^\perp) \mathcal{C} \oint \frac{d\sigma_3}{2\pi} \langle J[c_0^+ b_0^- V(\sigma_3) \Phi_B(x^\perp)] f_3[\Phi] \rangle. \tag{6.3}
\end{aligned}$$

The two equations (6.2) and (6.3) imply that the variation equation (3.12) is reduced to

$$\wp \left(\oint \frac{d\sigma_1}{2\pi} \Sigma_1[V(\sigma_1)] + \oint \frac{d\sigma_2}{2\pi} \Sigma_2[V(\sigma_2)] + \oint \frac{d\sigma_3}{2\pi} V(\sigma_3) \right) |B(x^\perp)\rangle = 0, \tag{6.4}$$

where we have used $|B(x^\perp)\rangle = c_0^+ b_0^- |\Phi_B(x^\perp)\rangle$. This is solved by the requirement

$$\Sigma_r[V(\sigma_r)] |B(x^\perp)\rangle = \frac{d}{d\sigma_r} \Sigma_r(\sigma_r) V(\Sigma_r(\sigma_r)) |B(x^\perp)\rangle. \tag{6.5}$$

With this condition, the integrations in Eq. (6.4) cancel, because the corresponding contours in the z -plane are C_1, C_2 and $-C_1 - C_2$ in Fig. 1, respectively.

Suppose the conformal mappings Σ_r are generic. Then, the infinitesimal form of Eq. (6.5) becomes

$$\mathcal{L}_n V(\sigma) |B(x^\perp)\rangle = e^{in\sigma} \left(-i \frac{d}{d\sigma} + n \right) V(\sigma) |B(x^\perp)\rangle. \tag{6.6}$$

This implies that $V(\sigma)$ must be a boundary primary operator with weight 1. This provides a sufficient condition to solve Eq. (3.12), and at the same time, it is identical to the physical state condition for the open string. We find that the entire *open* string spectrum is contained as a solution to Eq. (3.12), which is written in terms of the *closed* string variable. We note that the condition (6.6) implies the BRST invariance of the deformation, i.e.,

$$Q_B \oint \frac{d\sigma}{2\pi} V(\sigma) |B(x^\perp)\rangle = 0, \quad (6.7)$$

where the BRST operator Q_B is defined in Eq. (2.19). This can be proved using the identity

$$Q_B \mathcal{O}_{\text{matter}} |B(x^\perp)\rangle = \sum_{n=-\infty}^{\infty} c_{-n} \mathcal{L}_n \mathcal{O}_{\text{matter}} |B(x^\perp)\rangle. \quad (6.8)$$

6.2. Constraint from the interaction point

Actually, the mappings Σ_r are *not* generic but, rather, linear transformations with a discontinuity at the interaction point. In this sense, it is not obvious whether or not all the solutions to Eq. (6.4) are restricted to the boundary primary fields of dimension 1. Indeed, if we ignore the subtlety at the interaction point, the constraint from Eq. (6.5) does not imply that the operator $V(\sigma)$ must be *primary*, since the conformal transformation is restricted to linear transformations. In this sense, additional nontrivial constraints should come only from the interaction point. Instead of treating the generic vertex $V(\sigma)$, which would be technically difficult, we examine the lower excited mode explicitly. In particular, we pick the example of the massless vector particle. This is interesting, because it is difficult to confirm the transversality condition in the operator formalism,¹⁰⁾ due to the infinite-dimensionality of the Neumann coefficients. The computation involves a derivation of the *large* conformal transformation for the non-primary fields.

We first calculate the transformations of the scalar-type and vector-type operators V_S and V_V , which we considered in Ref. 10), and the modified version \hat{V}_V :³¹⁾

$$V_S(\sigma) = : e^{ik_\mu X^\mu(\sigma)} :, \quad V_V(\sigma) = : \zeta_\mu \partial_\sigma X^\mu(\sigma) e^{ik_\nu X^\nu(\sigma)} :, \quad (6.9)$$

$$\hat{V}_V(\sigma) \equiv V_V(\sigma) - (\zeta_\mu \theta^{\mu\nu} k_\nu / 4\pi) V_S(\sigma). \quad (6.10)$$

Here, $\theta \equiv \pi(\mathcal{O} - \mathcal{O}^T)/2 = -2\pi(1 + F)^{-1}F(1 - F)^{-1}$ is the noncommutativity parameter.²⁵⁾ For these operators, after a straightforward computation,³¹⁾ we obtain

$$\mathcal{L}_n V_S(\sigma) |B(x^\perp)\rangle = e^{in\sigma} (-i\partial_\sigma + n\Delta) V_S(\sigma) |B(x^\perp)\rangle, \quad (6.11)$$

$$\begin{aligned} \mathcal{L}_n \hat{V}_V(\sigma) |B(x^\perp)\rangle &= e^{in\sigma} (-i\partial_\sigma + n(\Delta + 1)) \hat{V}_V(\sigma) |B(x^\perp)\rangle \\ &+ e^{in\sigma} n^2 i \Xi V_S(\sigma) |B(x^\perp)\rangle, \end{aligned} \quad (6.12)$$

where we have defined $\Delta \equiv k_\mu G^{\mu\nu} k_\nu / 2$ and $\Xi \equiv -i\zeta_\mu G^{\mu\nu} k_\nu / 2$. The open string metric $G^{\mu\nu}$ appears due to the boundary state $|B(x^\perp)\rangle$ given in Eq. (3.1). These relations imply

$$\delta_\epsilon V_S(\sigma)|B(x^\perp)\rangle = \epsilon(\sigma)\partial_\sigma V_S(\sigma)|B(x^\perp)\rangle + \Delta\partial_\sigma\epsilon(\sigma)V_S(\sigma)|B(x^\perp)\rangle, \quad (6.13)$$

$$\begin{aligned} \delta_\epsilon \hat{V}_V(\sigma)|B(x^\perp)\rangle &= \epsilon(\sigma)\partial_\sigma \hat{V}_V(\sigma)|B(x^\perp)\rangle + (\Delta + 1)\partial_\sigma\epsilon(\sigma)\hat{V}_V(\sigma)|B(x^\perp)\rangle \\ &\quad + \Xi\partial_\sigma^2\epsilon(\sigma)V_S(\sigma)|B(x^\perp)\rangle \end{aligned} \quad (6.14)$$

for the infinitesimal transformation $\delta_\epsilon\sigma = -\sum_n \epsilon_n i e^{in\sigma} = \epsilon(\sigma)$. The last term in the second equation shows that \hat{V}_V is not primary unless $\Xi = 0$. A finite transformation should be determined from this infinitesimal deformation by the requirement of the cocycle condition, which states that under a sequence of coordinate transformations $\sigma_1 \rightarrow \sigma_2 \rightarrow \sigma_3$, the combination of two transformations $\sigma_1 \rightarrow \sigma_2$ followed by $\sigma_2 \rightarrow \sigma_3$ is identical to the direct one, $\sigma_1 \rightarrow \sigma_3$. The transformations of V_S and \hat{V}_T that satisfy this condition are

$$(d\sigma)^\Delta V_S(\sigma)|B(x^\perp)\rangle = (d\lambda)^\Delta V_S(\lambda)|B(x^\perp)\rangle, \quad (6.15)$$

$$(d\sigma)^{\Delta+1}\hat{V}_V(\sigma)|B(x^\perp)\rangle = (d\lambda)^{\Delta+1}\left(\hat{V}_V(\lambda)|B(x^\perp)\rangle - \Xi\frac{\partial_\lambda^2\sigma}{\partial_\lambda\sigma}V_S(\lambda)|B(x^\perp)\rangle\right). \quad (6.16)$$

The relevant formula is obtained by setting $\sigma = \sigma_r, \lambda = \sigma_3 = \Sigma_r(\sigma_r)$ (for $r = 1, 2$). Except at the interaction point, the anomalous term vanishes, since Σ_r is a linear function. Therefore, the only constraint for $V = \hat{V}_V$ is the on-shell condition, $\Delta = k_\mu G^{\mu\nu} k_\nu / 2 = 0$.

At the interaction point, the anomaly term diverges, due to the discontinuity of Σ_r . Geometrically, this results from the curvature singularity. In Fig. 4, we plot the landscape of the world-sheet at the interaction point. The three paths C_r are the contours of σ_r in Eq. (6.4), shifted slightly from the singularity. We use this shift to define the regularization.

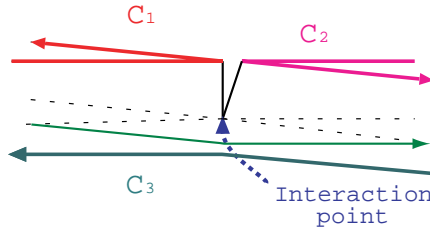


Fig. 4. Landscape near the interaction point.

If we impose the condition $\Delta = 0$, the contribution from most of the contour of integration mutually cancels. The only remaining part is the integration around the interaction point (see Fig. 5). At the interaction point, ρ does not give a proper parametrization, since $\arg(\rho)$ increases by 4π along the contour C_0 . It is therefore necessary to make a (large) transformation from ρ to z in the z -plane (see Fig. 1). We use $\sigma = \rho(z)/(i\alpha_3) + \text{const.}$

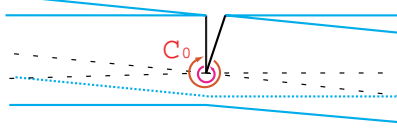


Fig. 5. The contour $C_1 + C_2 + C_3$ in Fig. 4 can be deformed into C_0 around the interaction point for \hat{V}_V with $\Delta = 0$.

[which corresponds to σ_3 , according to Eq. (2.13)] and $\lambda = z$ in Eq. (6.16) to evaluate the integration in Eq. (6.4). Noting that $d\rho/dz \sim \text{const.}(z - z_0)$ near the interaction point z_0 , we have

$$\begin{aligned} d\sigma \hat{V}_V(\sigma) |B(x^\perp)\rangle \\ = dz \left[\hat{V}_V(z) |B(x^\perp)\rangle - \Xi \left((z - z_0)^{-1} + \mathcal{O}((z - z_0)^0) \right) V_S(z) |B(x^\perp)\rangle \right] \end{aligned} \quad (6.17)$$

for $\Delta = 0$. Note that the anomalous term gives a pole residue. Therefore, the left-hand side of Eq. (6.4) for $\hat{V}_V(\sigma)$ is evaluated as

$$i\wp \Xi V_S(z_0) |B(x^\perp)\rangle = i\Xi \oint \frac{d\sigma}{2\pi} V_S(\sigma) |B(x^\perp)\rangle \quad (6.18)$$

for $\Delta = 0$. We conclude that not only the mass-shell condition $k_\mu G^{\mu\nu} k_\nu = 0$ but also the transversality condition $\Xi = \zeta_\mu G^{\mu\nu} k_\nu / 2 = 0$ should be imposed for the solution $\delta\Phi_B = \oint \frac{d\sigma}{2\pi} \hat{V}_V(\sigma) \Phi_B$ to Eq. (3.12).

§7. Summary and future problems

In this paper, we have derived some results which supplement our previous paper¹⁰⁾ with regard to the operator formulation. They include the determination of the analytic form of the coefficient of the idempotency relation and a generalization to the case of the Witten-type closed string vertex. The former implies that a large class of boundary states in the flat background satisfy a universal nonlinear relation.

We also used the representation of the HIKKO vertex in the CFT language and examined the idempotency relation in a background-independent fashion. There are some additional merits of this new computation. For example, we can show that the transversality condition of the massless vector particle is necessary. In the operator formalism, this is difficult, due to the appearance of the divergence.

We note that our proof in the case of a generic background is restricted to a confirmation of the weak condition. While this implies that the product of the boundary states is a linear combination of the boundary states, it does not fix the coefficients. In order to determine them, the Cardy condition is needed.

Because both the Cardy condition and the idempotency relation are quadratic in the string field, it is natural to conjecture that these two conditions are equivalent. For the flat background, this is indeed the case. The generic boundary states that satisfy the condition (5.1) are called Ishibashi states.³²⁾ For the flat background, they are given by the Fourier transformation of (3.1) with respect to x^\perp , and we denote them as $|p^\perp\rangle\rangle$. After the inclusion of the α parameter dependence and the insertion of the ghost zero modes, the star product of Ishibashi states is written

$$|p^\perp, \alpha_1\rangle\rangle \star |q^\perp, \alpha_2\rangle\rangle = \mathcal{C}c_0^+ |p^\perp + q^\perp, \alpha_1 + \alpha_2\rangle\rangle. \quad (7.1)$$

We note that the star product is not diagonal in this basis. It takes the form of the idempotency relation only when we take a linear combination (Fourier transformation) of Ishibashi states to form the Cardy states.

We conjecture that a relation similar to Eq. (7.1) holds for more general backgrounds. We write the generic Ishibashi state as $|i\rangle\rangle$, where i is the label of the highest weight representation. Then, a possible generalization of the above relation is

$$|i\rangle\rangle \star |j\rangle\rangle \propto \sum_k N_{ij}{}^k |k\rangle\rangle, \quad (7.2)$$

where $N_{ij}{}^k$ is Verlinde's fusion coefficient. By taking linear combinations of the form $|a\rangle = \sum_i \frac{S_{ia}}{\sqrt{S_{i0}}} |i\rangle\rangle$, where S_{ia} is the modular transformation matrix, the above relation is diagonalized with respect to the Cardy states $|a\rangle$:

$$|a\rangle \star |b\rangle \propto \delta_{ab} |b\rangle. \quad (7.3)$$

In the future, we will report our computation in the case of more generic backgrounds, including toroidal and orbifold compactification, to examine this conjecture.

As another important issue, we have observed that a VSFT-like scenario seems possible with the closed string variables. While this seems more hypothetical, there are some points that must be clarified in order to study this scenario more deeply. We have already commented that there seem to be no closed string physical states, even if we expand the action (3.16) around the boundary state. If this is true, our Lagrangian would seem to describe only open strings, while we have used closed string variables. In order to ascertain the consistency of this approach, the most nontrivial test would be to calculate the perturbative amplitudes such as the Veneziano amplitude. At that stage, we will need to clarify the nature of the α parameter. A natural interpretation may be that it is related to the moduli space of an open Riemann surface.

There are also various interesting directions in which our computation can be generalized. These include (i) a computation on a nontrivial background, which we have already mentioned, (ii) derivation of the noncommutative geometry from the closed string field theory, as we suggested in Footnote 6, (iii) a more detailed exploration of the properties of boundary states with respect to the star product of the Witten-type closed string field theory, in particular including the higher-order interactions, and finally (iv) the supersymmetric extension of the analysis, where the inclusion of the Ramond-Ramond sector is a major challenge in string field theory. The identity satisfied by the boundary states may provide information for the solution to this important problem.

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Appendix A

—— Notation and Conventions ——

In this section, we explicitly demonstrate the correspondence between the notation of HIKKO and of LPP. This is convenient for the purpose of comparing the results obtained in oscillator language with the original HIKKO notation (for example our previous paper¹⁰⁾) and those obtained in the LPP language.

First, the HIKKO oscillators $\alpha_n^{(+)}, \alpha_n^{(-)}, c_n^{(+)}, c_n^{(-)}, \bar{c}_n^{(+)}, \bar{c}_n^{(-)}$ correspond to the LPP oscillators (or in the conventional CFT notation) $\alpha_n, \tilde{\alpha}_n, c_n, \tilde{c}_n, b_n, \tilde{b}_n$, respectively. In particular, the ghost zero modes are related as

$$\begin{aligned} \left(\frac{\partial}{\partial \bar{c}_0} = \frac{1}{2}(c_0^{(+)} + c_0^{(-)}) \right)_{\text{H}} &= \left(c_0^+ = \frac{1}{2}(c_0 + \tilde{c}_0) \right)_{\text{L}}, \\ \left(i \frac{\partial}{\partial \pi_c^0} = c_0^{(+)} - c_0^{(-)} \right)_{\text{H}} &= (c_0^- = c_0 - \tilde{c}_0)_{\text{L}}, \\ \left(\bar{c}_0 = \bar{c}_0^{(+)} + \bar{c}_0^{(-)} \right)_{\text{H}} &= (b_0^+ = b_0 + \tilde{b}_0)_{\text{L}}, \\ \left(-i\pi_c^0 = \frac{1}{2}(\bar{c}_0^{(+)} - \bar{c}_0^{(-)}) \right)_{\text{H}} &= \left(b_0^- = \frac{1}{2}(b_0 - \tilde{b}_0) \right)_{\text{L}}. \end{aligned} \tag{A.1}$$

We often indicate a quantity in the original HIKKO notation by the subscript H and a

quantity in the LPP notation by the subscript L in this section. (In other sections, we have used the LPP notation and did not include the subscript L.)

In bra-ket notation, we identify the vacuum convention by

$${}_H\langle 0| = {}_L\langle 1, 1|c_0^-c_0^+, \quad |0\rangle_H = |1, 1\rangle_L, \quad (\text{A}\cdot 2)$$

where ${}_L\langle 1, 1| := \langle 0|c_{-1}\tilde{c}_{-1}$, $|1, 1\rangle_L = c_1\tilde{c}_1|0\rangle$, and $\langle 0|$ and $|0\rangle$ denote conformal vacua and are normalized as $\langle 1, 1|c_0\tilde{c}_0|1, 1\rangle = \langle 1, 1|c_0^-c_0^+|1, 1\rangle = (2\pi)^d\delta^d(0) = V_d$. The string field in the LPP formulation can be obtained from that in the HIKKO formulation by Fourier transformation with respect to ghost zero mode:

$$-i \int d\bar{c}_0 d\pi_c^0 e^{-\bar{c}_0 c_0^+ + i\pi_c^0 c_0^-} |\Phi(\alpha)\rangle_H = |\Phi(\alpha)\rangle_L. \quad (\text{A}\cdot 3)$$

This implies that the ghost zero mode expansion becomes

$$\begin{aligned} & -i \int d\bar{c}_0 d\pi_c^0 e^{-\bar{c}_0 c_0^+ + i\pi_c^0 c_0^-} \left[-\bar{c}_0|\phi\rangle + |\psi\rangle + \bar{c}_0\pi_c^0|\chi\rangle + i\pi_c^0|\eta\rangle \right]_H \\ & = \left[c_0^-|\phi\rangle + c_0^-c_0^+|\psi\rangle + i|\chi\rangle - c_0^+|\eta\rangle \right]_L. \end{aligned} \quad (\text{A}\cdot 4)$$

As expected, the physical sector $\bar{c}_0|\phi\rangle$ in the HIKKO formulation given in Ref. 11) is mapped to $c_0^-|\phi\rangle$ in the LPP convention.

A.1. Reflector

The reflector in the HIKKO theory¹¹⁾ is given by

$$\begin{aligned} \langle \tilde{R}(1, 2)| &= \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \frac{d\alpha_1}{2\pi} \frac{d\alpha_2}{2\pi} \langle p_1, \alpha_1 | \langle p_2, \alpha_2 | \\ &\times e^{-\sum_{\pm, n \geq 1} (a_n^{(\pm)(1)} a_n^{(\pm)(2)} + c_n^{(\pm)(1)} \bar{c}_n^{(\pm)(2)} - \bar{c}_n^{(\pm)(1)} c_n^{(\pm)(2)})} \\ &\times (2\pi)^d \delta^d(p_1 + p_2) \delta(\pi_c^{0(1)} - \pi_c^{0(2)}) \delta(\bar{c}_0^{(1)} - \bar{c}_0^{(2)}) 2\pi \delta(\alpha_1 + \alpha_2), \end{aligned} \quad (\text{A}\cdot 5)$$

where we have used the momentum representation for the matter zero mode. We adopted the bra-ket notation for the p_μ, α part, in addition to the nonzero mode oscillator sector. We can obtain the bra from the ket using the reflector (A·5): ${}_2\langle \Phi(-\alpha)| = \int d\bar{c}_0^{(1)} d\pi_c^{0(1)} \langle \tilde{R}(1, 2)|\Phi(\alpha)\rangle_1$. In particular, we need to carry out the zero modes integration in this convention. The dot product of string fields is defined with the “metric” π_c^0 :

$$\begin{aligned} \Phi_1 \cdot \Phi_2 &= \int d\bar{c}_0^{(1)} d\pi_c^{0(1)} d\bar{c}_0^{(2)} d\pi_c^{0(2)} \langle \tilde{R}(2, 1)|\pi_c^{0(1)}|\Phi_1\rangle_1 |\Phi_2\rangle_2 \\ &= (-1)^{|\Phi_1||\Phi_2|} \Phi_2 \cdot \Phi_1. \end{aligned} \quad (\text{A}\cdot 6)$$

Next, we review the LPP reflector and relate it to the HIKKO reflector. It is a kind of two-string vertex and therefore is determined by fixing two conformal mappings. We take $h_1(z) = I(z) := 1/z$ and $h_2(z) = z$ as conformal mappings:

$$\begin{aligned} \langle R(1, 2) | &= \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \langle p_1, 1, 1 | \langle p_2, 1, 1 | \\ &\times e^{-\sum_{n \geq 1} (a_n^{(1)} a_n^{(2)} + c_n^{(1)} b_n^{(2)} - b_n^{(1)} c_n^{(2)} + \tilde{a}_n^{(1)} \tilde{a}_n^{(2)} + \tilde{c}_n^{(1)} \tilde{b}_n^{(2)} - \tilde{b}_n^{(1)} \tilde{c}_n^{(2)})} \\ &\times (2\pi)^d \delta^d(p_1 + p_2) (c_0^{-(1)} + c_0^{-(2)}) (c_0^{+(1)} + c_0^{+(2)}). \end{aligned} \quad (\text{A} \cdot 7)$$

Here, we have slightly rewritten the ghost zero mode part given in Ref. 12) using the relation $\langle 3, 3 | = \langle 1, 1 | c_0 \tilde{c}_0 c_1 \tilde{c}_1$. The BPZ conjugate is obtained using this reflector, ${}_2\langle \Phi | = \langle R(1, 2) | \Phi \rangle_1$, and the BPZ inner product becomes $\langle \Phi_1, \Phi_2 \rangle = \langle R(1, 2) | \Phi_1 \rangle_1 | \Phi_2 \rangle_2 = \langle I[\Phi_1] | \Phi_2 \rangle$. We can rewrite the inner product of the LPP string fields Φ_{1L} and Φ_{2L} with the “metric” b_0^- in terms of that in the HIKKO formulation using the correspondence given by Eqs. (A·2) and (A·3) as

$$\begin{aligned} -\langle \Phi_1, b_0^- \Phi_2 \rangle &= -\langle R(1, 2) | \Phi_1 \rangle_{1L} b_0^{-(2)} | \Phi_2 \rangle_{2L} \\ &= \int d\bar{c}_0^{(1)} d\pi_c^{0(1)} d\bar{c}_0^{(2)} d\pi_c^{0(2)} \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \\ &\times {}_{1H} \langle p_1 | {}_{2H} \langle p_2 | (\pi_c^{0(1)} - \pi_c^{0(2)}) (\bar{c}_0^{(1)} - \bar{c}_0^{(2)}) \\ &\times (2\pi)^d \delta^d(p_1 + p_2) | \Phi_1 \rangle_{1H} \pi_c^{0(2)} | \Phi_2 \rangle_{2H}. \end{aligned} \quad (\text{A} \cdot 8)$$

Thus, including the α sector, we have obtained the dot product formula as given by Eq. (2·4).

Finally, we note that in this paper, we include π_c^0 ; that is, we do not use the so-called π_c^0 -omitted formulation, as in Refs. 10) and 33).

A.2. 3-string vertex and Neumann coefficients

The HIKKO 3-string vertex^{11),35)} and the LPP 3-string vertex¹²⁾ are equivalent under the above correspondence. We will briefly demonstrate this fact. (See Refs. 34) and 36) for details.)

First, we rewrite the 3-string vertex given as a ket¹¹⁾ into the form of a bra:

$$\begin{aligned} \langle V(1, 2, 3) | &:= \int d\bar{c}_0^{(1')} d\pi_c^{0(1')} d\bar{c}_0^{(2')} d\pi_c^{0(2')} d\bar{c}_0^{(3')} d\pi_c^{0(3')} \langle \tilde{R}(1, 1') | \langle \tilde{R}(2, 2') | \langle \tilde{R}(3, 3') | V(1', 2', 3') \rangle. \end{aligned}$$

Then, multiplying by $\pi_c^{0(3)}$ from the right and noting the identity $\frac{\alpha_1 \alpha_2}{\alpha_3} \Pi_c \delta \left(\sum_{r=1}^3 \alpha_r^{-1} \pi_c^{0(r)} \right) \pi_c^{0(3)} = \pi_c^{0(1)} \pi_c^{0(2)} \pi_c^{0(3)}$, with $\alpha_1 + \alpha_2 + \alpha_3 = 0$, we have

$$\langle V(1, 2, 3) | \pi_c^{0(3)} = \int \delta(1, 2, 3) [\mu(1, 2, 3)]^2 \langle p_1, \alpha_1 | \langle p_2, \alpha_2 | \langle p_3, \alpha_3 | e^{E_H(1, 2, 3)}$$

$$\times \pi_c^{0(1)} \pi_c^{0(2)} \pi_c^{0(3)} \wp^{(1)} \wp^{(2)} \wp^{(3)}, \quad (\text{A}\cdot 9)$$

$$E_H(1, 2, 3) = \frac{1}{2} \sum_{\pm, r, s} \sum_{n, m=0}^{\infty} \alpha_n^{(\pm)(r)} \bar{N}_{nm}^{rs} \alpha_m^{(\pm)(s)} - \sum_{\pm, r, s} \sum_{n, m=1}^{\infty} c_n^{(\pm)(r)} n \frac{\alpha_r}{\alpha_s} \bar{N}_{nm}^{rs} \bar{c}_m^{(\pm)(s)} \\ - \frac{1}{2} \sum_{\pm, r, s} \sum_{n=1}^{\infty} c_n^{(\pm)(r)} (\alpha_r w_n^{sr}) \bar{c}_0^{(s)}, \quad (\text{A}\cdot 10)$$

$$\mu(1, 2, 3) = e^{-\tau_0 \sum_{r=1}^3 \alpha_r^{-1}}, \quad \tau_0 = \sum_{r=1}^3 \alpha_r \log |\alpha_r|, \quad (\text{A}\cdot 11)$$

$$\int \delta(1, 2, 3) = \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \frac{d^d p_3}{(2\pi)^d} \frac{d\alpha_1}{2\pi} \frac{d\alpha_2}{2\pi} \frac{d\alpha_3}{2\pi} \\ \times (2\pi)^d \delta^d(p_1 + p_2 + p_3) 2\pi \delta(\alpha_1 + \alpha_2 + \alpha_3), \quad (\text{A}\cdot 12)$$

$$w_m^{rs} = \alpha_r^{-1} \left[\delta_{r,s} \cos m \sigma_I^{(r)} - m \sum_{n=1}^{\infty} \bar{N}_{mn}^{sr} \cos n \sigma_I^{(r)} \right]. \quad (\text{A}\cdot 13)$$

Here, $\sigma_I^{(r)}$ is the interaction point of the string r , and $\wp^{(r)}$ is the projection operator for the level matching condition of this string, given in Eq. (2.11). The Neumann coefficients are given by¹¹

$$\bar{N}_{nm}^{rs} = \frac{1}{nm} \oint_0 \frac{dw_r}{2\pi i} \oint_0 \frac{dw_s}{2\pi i} h'_r(w_r) h'_s(w_s) \frac{w_r^{-n} w_s^{-m}}{(h_r(w_r) - h_s(w_s))^2}, \quad (n, m \geq 1) \quad (\text{A}\cdot 14)$$

$$\bar{N}_{n0}^{rs} = \bar{N}_{0n}^{sr} = \frac{1}{n} \oint_0 \frac{dw_r}{2\pi i} h'_r(w_r) \frac{w_r^{-n}}{h_r(w_r) - h_s(0)}, \quad (n \geq 1) \quad (\text{A}\cdot 15)$$

$$\bar{N}_{00}^{rs} = \log |Z_r - Z_s| = \log |h_r(0) - h_s(0)|, \quad (r \neq s) \quad (\text{A}\cdot 16)$$

$$\bar{N}_{00}^{rr} = - \sum_{i \neq r} \frac{\alpha_i}{\alpha_r} \log |Z_r - Z_s| + \frac{1}{\alpha_r} \tau_0(Z_1, Z_2, Z_3) = \log |h'_r(0)|, \quad (\text{A}\cdot 17)$$

where $z = h_r(w_r)$ (with $|w_r| \leq 1$), which is a gluing map of the string r into the z -plane, is defined by $\rho(z) = \alpha_r \log w_r + \tau_0(Z_1, Z_2, Z_3) + i\pi \sum_{s=1}^{r-1} \alpha_s$, with the Mandelstam mapping $\rho(z) = \sum_{r=1}^3 \alpha_r \log(z - Z_r)$. Here, $Z_r = h_r(0)$ and $\tau_0(Z_1, Z_2, Z_3) = \text{Re} \rho(z_0)$, where z_0 is the interaction point, where we have $\rho'(z_0) = 0$. We note that the Neumann coefficients of the ghost sector are related to those of Kunitomo-Suehiro vertex²⁰⁾ in the $P = 1$ picture as $-n(\alpha_r/\alpha_s) \bar{N}_{nm}^{rs} = N_{nm}^{(1)rs}$, $-\alpha_r w_n^{sr} = N_{n0}^{(1)rs}$ ($n, m \geq 1$).

Next, we rewrite the LPP 3-string vertex for the conformal mappings $h_r(w_r)$ ($r = 1, 2, 3$), which is given by Eq. (2.1). In explicit form using oscillators, the Neumann coefficients in

¹¹ We can choose the three real parameters Z_1, Z_2 and Z_3 arbitrarily as long as p_r and α_r are conserved. They are often chosen as $Z_1 = 1, Z_2 = 0$ and $Z_3 = \infty$ [with a constant shift in $\rho(z)$] for convenience.³⁵⁾ Though the Neumann coefficients in the anti-holomorphic sector are the complex conjugates of those in the holomorphic sector in general, those of the 3-string vertex for both the light-cone and the Witten type are real.

the matter sector are the same as those of the HIKKO, \bar{N}_{nm}^{rs} . The Neumann coefficients in the ghost sector are given by¹²⁾

$$N_{nm}^{(gh)rs} = \oint_0 \frac{dw_r}{2\pi i} \oint_0 \frac{dw_s}{2\pi i} (h'_r(w_r))^2 (h'_s(w_s))^{-1} \frac{-w_r^{-n+1} w_s^{-m-2}}{h_r(w_r) - h_s(w_s)}, \quad (A.18)$$

$$(n \geq 2, m \geq -1)$$

$$M_{in}^r = \oint_0 \frac{dw_r}{2\pi i} (h'_r(w_r))^{-1} w_r^{-n-2} (h_r(w_r))^{i+1}. \quad (n \geq -1, i = -1, 0, 1) \quad (A.19)$$

They satisfy the following identities:

$$M_{i,-1}^r = Z_r^{i+1} e^{-\bar{N}_{00}^{rr}}, \quad (A.20)$$

$$\det_{r,i} M_{i,-1}^r = |Z_1 - Z_2| |Z_2 - Z_3| |Z_3 - Z_1| e^{-\sum_{r=1}^3 \bar{N}_{00}^{rr}} = \mu(1, 2, 3), \quad (A.21)$$

$$N_{nm}^{(1)rs} = N_{nm}^{(gh)rs} - \sum_{t,i} N_{n,-1}^{(gh)rt} ((M_{,-1})^{-1})_{ti} M_{im}^s, \quad (n \geq 2, m \geq 0) \quad (A.22)$$

$$N_{1m}^{(1)rs} = \sum_i ((M_{,-1})^{-1})_{ri} M_{im}^s, \quad (m \geq 0) \quad (A.23)$$

$$\begin{aligned} & {}_1\langle 3, 3| {}_2\langle 3, 3| {}_3\langle 3, 3| \prod_{i=-1}^1 \left(\sum_r \sum_{m \geq -1} M_{im}^r b_m^{(r)} \right) \prod_{i=-1}^1 \left(\sum_r \sum_{m \geq -1} M_{im}^r \tilde{b}_m^{(r)} \right) \\ &= {}_1\langle 1, 1| c_0^{-(1)} c_0^{+(1)} {}_2\langle 1, 1| c_0^{-(2)} c_0^{+(2)} {}_3\langle 1, 1| c_0^{-(3)} c_0^{+(3)} \\ &\quad \times e^{\sum_{r,s} \sum_{m \geq 0} (c_1^{(r)} N_{1m}^{(1)rs} b_m^{(s)} + \tilde{c}_1^{(r)} N_{1m}^{(1)rs} \tilde{b}_m^{(s)})} [\mu(1, 2, 3)]^2. \end{aligned} \quad (A.24)$$

Using these relations, we obtain the LPP vertex as follows:

$$\begin{aligned} \langle v_3| &= \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \frac{d^d p_3}{(2\pi)^d} \\ &\quad \times {}_1\langle p_1, 1, 1| c_0^{-(1)} c_0^{+(1)} {}_2\langle p_2, 1, 1| c_0^{-(2)} c_0^{+(2)} {}_3\langle p_3, 1, 1| c_0^{-(3)} c_0^{+(3)} \\ &\quad \times (2\pi)^d \delta^d(p_1 + p_2 + p_3) e^{E_{KS}(1,2,3)} [\mu(1, 2, 3)]^2, \end{aligned} \quad (A.25)$$

$$\begin{aligned} E_{KS}(1, 2, 3) &= \frac{1}{2} \sum_{r,s} \sum_{n,m=0}^{\infty} \alpha_n^{(r)} \bar{N}_{nm}^{rs} \alpha_m^{(s)} + \frac{1}{2} \sum_{r,s} \sum_{n,m=0}^{\infty} \tilde{\alpha}_n^{(r)} \bar{N}_{nm}^{rs} \tilde{\alpha}_m^{(s)} \\ &\quad + \sum_{r,s} \sum_{n \geq 1, m \geq 0} c_n^{(r)} N_{nm}^{(1)rs} b_m^{(s)} + \sum_{r,s} \sum_{n \geq 1, m \geq 0} \tilde{c}_n^{(r)} N_{nm}^{(1)rs} \tilde{b}_m^{(s)}. \end{aligned} \quad (A.26)$$

Finally, noting the relations (A.3), (A.9), (A.25) and $\langle v_3| e^{\sum_r c_0^{+(r)} \tilde{c}_0^{(r)}} = \langle v_3|_{b_0^{(r)}, \tilde{b}_0^{(r)} \rightarrow \frac{1}{2} \tilde{c}_0^{(r)}}$, we have obtained the correspondence between the HIKKO \star product and the LPP 3-string vertex for the string fields, $|\Phi_r(\alpha_r)\rangle = |\Phi_r\rangle \otimes |\alpha_r\rangle$:

$$\begin{aligned} & (\Phi_1(\alpha_1) \star \Phi_2(\alpha_2)) \cdot \Phi_3(\alpha_3) \\ &= \int d\bar{c}_0^{(1)} d\pi_c^{0(1)} d\bar{c}_0^{(2)} d\pi_c^{0(2)} d\bar{c}_0^{(3)} d\pi_c^{0(3)} \langle V(1, 2, 3) | \pi_c^{0(3)} | \Phi_1(\alpha_1) \rangle_{1H} | \Phi_2(\alpha_2) \rangle_{2H} | \Phi_3(\alpha_3) \rangle_{3H} \end{aligned}$$

$$\begin{aligned}
&= - \int \frac{d\alpha'_1}{2\pi} \frac{d\alpha'_2}{2\pi} \frac{d\alpha'_3}{2\pi} {}_1\langle\alpha'_1| {}_2\langle\alpha'_2| {}_3\langle\alpha'_3| 2\pi\delta(\alpha'_1 + \alpha'_2 + \alpha'_3) \\
&\quad \times \langle v_3| b_0^{-(1)} b_0^{-(2)} b_0^{-(3)} \wp^{(1)} \wp^{(2)} \wp^{(3)} | \Phi_1(\alpha_1) \rangle_{1L} | \Phi_2(\alpha_2) \rangle_{2L} | \Phi_3(\alpha_3) \rangle_{3L} \\
&= -2\pi\delta(\alpha_1 + \alpha_2 + \alpha_3) (-1)^{|\Phi_2|} \langle h_1[b_0^- \wp \Phi_1] h_2[b_0^- \wp \Phi_2] h_3[b_0^- \wp \Phi_3] \rangle. \tag{A.27}
\end{aligned}$$

Appendix B

— Computation of the Determinant —

A nontrivial divergent factor \mathcal{C} appears in (3.5). In this section, we compute it analytically using the so-called Cremmer–Gervais identity.¹⁸⁾ This identity provides us a method to compute 4-string amplitudes constructed from 3-string vertices. Interestingly enough, in the procedure to evaluate 4-string amplitudes, we encounter a determinant of Neumann coefficient matrices that is identical to the factor that we find in the computation of the left-hand side of Eq. (3.5). The Cremmer–Gervais identity allows us to regularize the divergent factor \mathcal{C} . This regularization corresponds to introducing the propagation of an intermediate string in the 4-string vertex and reveals its dependence on α .

The factor that we would like to consider is

$$\mathcal{C} = \mu(1, 2, 3)^2 (\det(1 - r^2))^{-\frac{d-2}{2}}, \tag{B.1}$$

where $\mu(1, 2, 3)$ is given in Eq. (A.11), and the matrix r is

$$r_{mn} = \frac{\beta(\beta+1)(mn)^{3/2}}{m+n} \bar{f}_m^{(3)} \bar{f}_n^{(3)}, \quad \bar{f}_m^{(3)} = \frac{\Gamma(-m\beta) e^{m(\beta \log |\beta| - (\beta+1) \log |\beta+1|)}}{m! \Gamma(-m\beta + 1 - m)}, \tag{B.2}$$

with $\beta = \alpha_1/\alpha_3$, $\alpha_3 = -\alpha_1 - \alpha_2$.¹⁰⁾ We note that this matrix r is given by the Neumann coefficients for the 3-string vertex: $r_{mn} = \sqrt{mn} \bar{N}_{mn}^{33}(\alpha_1, \alpha_2, \alpha_3) =: \tilde{N}_{mn}^{33}$.

The Cremmer–Gervais identity applied to our case¹²⁾ (which is a scattering process, such as that depicted in Fig. 6) is

$$\frac{1}{|\alpha_3|} \left(\det \left(1 - \tilde{N}^{33} \tilde{N}_T^{33} \right) \right)^{-12} e^{\frac{T}{-\alpha_3}} = \exp \left(-\log \left(Z_3^2 \frac{\partial T}{\partial Z_3} \right) - \alpha_3^2 \mathcal{A} + \frac{2\tau_0}{\alpha_3} \right). \tag{B.3}$$

On the left-hand side, the matrix \tilde{N}_T^{33} is given by the Neumann coefficient for the 3-string vertex:

$$\begin{aligned}
\tilde{N}_{Tmn}^{33} &= \sqrt{mn} \bar{N}_{mn}^{33}(\alpha_1, \alpha_2, \alpha_3) e^{\frac{(m+n)T}{\alpha_3}} \\
&= (-1)^{m+n} \sqrt{mn} \bar{N}_{mn}^{33}(-\alpha_2, -\alpha_1, -\alpha_3) e^{\frac{(m+n)T}{\alpha_3}}. \tag{B.4}
\end{aligned}$$

¹²⁾ Here, we have used equations given in Appendix C of Ref. 11) with $(\alpha_3, \alpha_4) \rightarrow (-\alpha_2, -\alpha_1)$ and $\theta = 0$. Our \mathcal{A} corresponds to the quantity $a + c$ there.

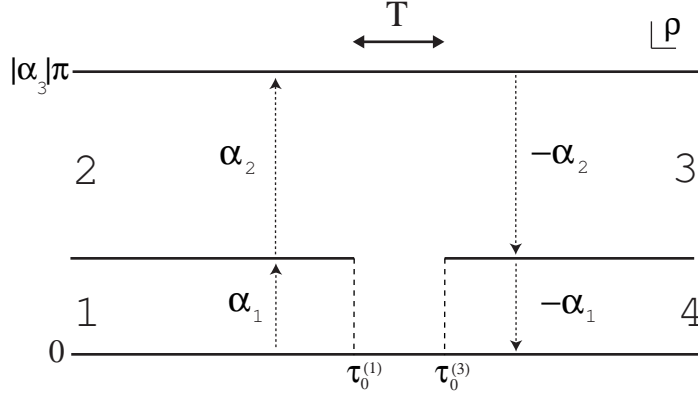


Fig. 6. The 4-string configuration to obtain our determinant. Here, we display only the $\text{Im } \rho \geq 0$ region for simplicity. The interval T corresponds to our regularization parameter. We consider the case in which $\alpha_1, \alpha_2 > 0$.

Here, $\alpha_3 = -\alpha_1 - \alpha_2$, and \tilde{N}_{mn}^{33} is the $T \rightarrow 0$ limit of \tilde{N}_{Tmn}^{33} . On the right-hand side of Eq. (B.3), we have $\tau_0 = \alpha_1 \log |\alpha_1| + \alpha_2 \log |\alpha_2| + \alpha_3 \log |\alpha_3|$, and the quantity \mathcal{A} is given by the Neumann coefficients for the 4-string vertex that corresponds to Fig. 6:

$$\alpha_3^2 \mathcal{A} = \text{Re} \left(\sum_{r=1}^4 \bar{N}_{00}^{(4)rr} - (\bar{N}_{00}^{(4)12} + \bar{N}_{00}^{(4)21} + \bar{N}_{00}^{(4)34} + \bar{N}_{00}^{(4)43}) - 2\tau_0 \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right) \right), \quad (\text{B.5})$$

$$\text{Re } \bar{N}_{00}^{(4)rs} = \begin{cases} \log |Z_r - Z_s|, & r \neq s, \\ -\sum_{i \neq r} \frac{\alpha_i}{\alpha_r} \log |Z_r - Z_i| + \frac{1}{\alpha_r} \tau_0^{(r)}, & r = s, \end{cases} \quad (\text{B.6})$$

$$\tau_0^{(1)} = \tau_0^{(2)} = \text{Re } \rho(z_-), \quad \tau_0^{(3)} = \tau_0^{(4)} = \text{Re } \rho(z_+). \quad (\text{B.7})$$

Here, the interaction points z_{\pm} are the two solutions of $\frac{d\rho(z)}{dz} = 0$, where $\rho(z)$ is the Mandelstam mapping defined as

$$\rho(z) = \alpha_1 \log(z - Z_1) + \alpha_2 \log(z - Z_2) - \alpha_2 \log(z - Z_3) - \alpha_1 \log(z - Z_4). \quad (\text{B.8})$$

We fix the gauge as $Z_1 = \infty, Z_2 = 1 > Z_3 > Z_4 = 0$. Then we obtain

$$z_{\pm} = -(2\alpha_1)^{-1} \left(\alpha_3 + (\alpha_2 - \alpha_1)Z_3 \pm \Delta^{\frac{1}{2}} \right), \quad (\text{B.9})$$

$$\Delta = (\alpha_1 + \alpha_2)^2 (1 - Z_3) \{1 - (2\beta + 1)^2 Z_3\}, \quad (\text{B.10})$$

with $-1 < \beta = \alpha_1/\alpha_3 < 0$. The time interval T that represents the propagation of the intermediate string is given by

$$T = \tau_0^{(3)} - \tau_0^{(1)} = \text{Re}(\rho(z_+) - \rho(z_-)), \quad (\text{B.11})$$

and it is a function of Z_3 through z_{\pm} .

Now, we regularize the factor \mathcal{C} , or r with the parameter T so that we use Eq. (B.3),

$$(r^2)_{nm} \rightarrow (rr_T)_{nm}, \quad (r_T)_{nm} := e^{-(n+m)\frac{T}{\alpha_1+\alpha_2}} r_{nm}. \quad (\text{B.12})$$

Plugging this into Eq. (B.3), we obtain the regularized expression of the factor:

$$\begin{aligned} & \mu(1, 2, 3)^2 (\det(1 - rr_T))^{-12} \\ &= \exp \left(\frac{\beta^2 + \beta + 1}{\beta(\beta + 1)} \frac{T}{\alpha_3} - (1 + \beta + \beta^2) \log Z_3 \right. \\ & \quad \left. - \frac{1}{2} \log(1 - (2\beta + 1)^2 Z_3) - \frac{3}{2} \log(1 - Z_3) \right). \end{aligned} \quad (\text{B.13})$$

The right-hand side of this expression gives the factor (B.1) in the limit $T \rightarrow 0$ for the critical dimension, $d = 26$. It can be seen from Eqs. (B.11), (B.9) and (B.10) that this corresponds to taking $z_+ \rightarrow z_-$, and consequently $Z_3 \rightarrow 1$. If we define $\varepsilon \equiv 1 - Z_3$, then T can be expressed as

$$\frac{T}{|\alpha_3|} = 4\varepsilon^{\frac{1}{2}} \sqrt{-\beta(\beta + 1)} + \mathcal{O}(\varepsilon). \quad (\text{B.14})$$

In the limit that $Z_3 \rightarrow 1$, Eq. (B.13) reduces to

$$\begin{aligned} \mu(1, 2, 3)^2 (\det(1 - rr_T))^{-12} &= 2^5 \left(\frac{T}{|\alpha_3|} \right)^{-3} [-\beta(\beta + 1)] \{1 + \mathcal{O}(T)\} \\ &\rightarrow 2^5 T^{-3} |\alpha_1 \alpha_2 (\alpha_1 + \alpha_2)|. \end{aligned} \quad (\text{B.15})$$

The regularization that we adopted here for evaluation of the factor \mathcal{C} , i.e. Eq. (B.1), is consistent with the level truncation approximation if we make the identification $T^{-1} \sim L$, where L is the size of the truncated matrix r (B.2), or Neumann matrix \bar{N}^{33} . Although we observed that $\mathcal{C} \propto L^3$ in Ref. 10), we further investigated its β dependence in the case of the critical dimension, $d = 26$. In fact, we have plotted $\mathcal{C}/(L^3(-\beta)(\beta + 1))$ up to $L = 2000$ using the computer program *Mathematica5* and confirmed its convergence to a constant (~ 772) independent of β (see Fig. 7). This numerical result implies that our regularization of \mathcal{C} , Eq. (B.15), is consistent with the level truncation of $\bar{N}_{mn}^{33}(\alpha_1, \alpha_2, \alpha_3)$ through the identification of the parameters $L \sim |\alpha_3|/T$.

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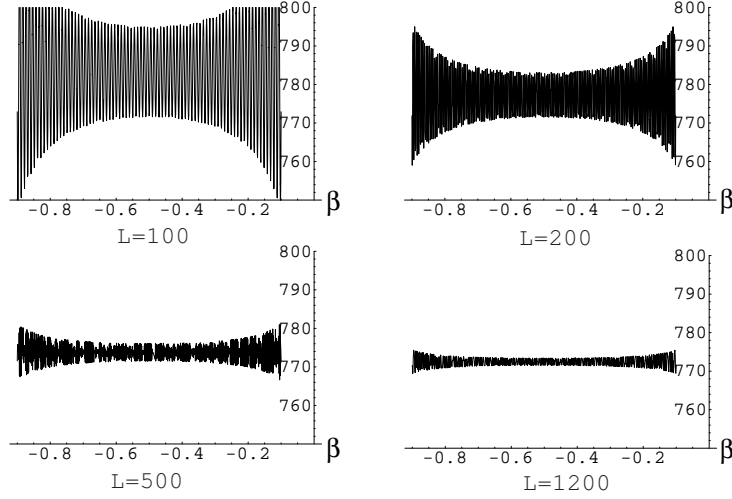


Fig. 7. Plots of $C/(L^3(-\beta)(\beta+1))$ for $L = 100, 200, 500, 1200$ in the interval $-0.9 \leq \beta \leq -0.1$ obtained using *Mathematica5*. Although this quantity oscillates as a function of β for small L , it converges to a constant independent of β as $L \rightarrow \infty$.

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